Jesse Gage

Math 230A

Dr. Zhou

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Homework: Extra Credit #1

16. *Regard* $\mathbb{Q}$*, the set of all rational numbers, as a metric space, with* $d(p,q) = |p-q|$*. Let E be the set of all* $p \in \mathbb{Q}$ *such that* $2 < p^2 < 3$*. Show that E is closed and bounded in* $\mathbb{Q}$*, but that E is not compact. Is E open in* $\mathbb{Q}$*?*

Note that E consists of the rational numbers in $\left(-\sqrt{3}, -\sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{3}\right)$. Clearly, E is bounded by $(-2,2)$. Furthermore, E is closed in $\mathbb{Q}$ (not in $\mathbb{R}$!) because $\mathbb{Q}$ is dense in $\mathbb{R}$, which means every limit point of $\mathbb{Q}$ is in $\mathbb{Q}$.

However, E is not compact. Suppose we have a cover where $H_1 = \left(-\sqrt{3}, -\sqrt{2}\right)$, $H_2 = \left(\sqrt{2}, \frac{\pi}{2}\right)$ and $G_\alpha = \frac{\pi}{2} + \frac{1}{\alpha}$ for $\alpha \in \mathbb{N}$. Then the collection of sets

$$G = H_1 \cup H_2 \cup \bigcup_{\alpha \in \mathbb{N}} G_\alpha$$

forms a cover of E, but from earlier homework we G cannot be made finite because we can't throw away all but a finite number of G_α's.

Is E open in $\mathbb{Q}$? Yes. Let $p \in \mathbb{Q}$. Then there is a neighborhood $N_r(p) \subseteq E$ (remember $E \subseteq \mathbb{Q}$ so there are no irrationals in N), where r is less than minimum of the distance (in $\mathbb{R}$) from p to $\left\{\pm\sqrt{3}, \pm\sqrt{2}\right\}$.

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17. *Let E be the set of all* $x \in [0,1]$ *whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in* $[0,1]$*? Is E compact? Is E perfect?*

Note that

$$E = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{10^n} \right\},$$

where each a_i is either 4 or 7.

E is not countable, as it can easily be put into one-to-one correspondence with the set of 0-1 sequences, which are countable.

E is very clearly not dense in $[0,1]$, because the entire range of $\left(0.4\overline{7}, 0.7\overline{4}\right)$ contains no points of E; 0.5 for example is thus not a limit point.

E is, however, compact. Boundedness is more or less a gimme, as $E \subseteq \left[0.\overline{4}, 0.\overline{7}\right]$.

We need to show that E is closed. Let p be a limit point of E, and suppose that $p \notin E$. Write the decimal expansion of p as

$$p = 0.p_1 p_2 \ldots p_n \ldots.$$

Since $p \notin E$, there are some digits in p that are not 4 or 7; let p_k be the leftmost such digit.

- If $p_k \in \{0,1,2,3\}$, then some $p_1, p_2, \ldots$ prior to p_k must be 7; if not then $p < 0.444\ldots$ and we could find a neighborhood around p that contains no points of E. Let p_m be the leftmost 7. Then

  $$0.p_1p_2\ldots p_{m-1}4p_{m+1}\ldots p_{k-1}777\ldots < p < 0.p_1\ldots p_{k-1}444...$$

  which again is a neighborhood around p that has no elements of E. Thus this case yields a contradiction.

- If $p_k \in \{5,6\}$ then

  $$0.p_1\ldots p_{k-1}4777\ldots < p < 0.p_1\ldots p_{k-1}7444\ldots$$

  which is neighborhood of p with no elements of E; this is a contradiction.

- The case where $p_k \in \{8,9\}$ is similar to the first case; there is some p_m prior to $p_k$that must be 4 or else $0.77777\ldots < p$. Then

  $$0.p_1p_2\ldots p_{k-1}7 < p < 0.p_1\ldots p_{m-1}7p_{m+1}\ldots$$

  which is another neighborhood around p with no points in E and another contradiction.

Thus $p_k \notin \{4,7\}$ yields a contradiction. Thus $p \in E$ so E is closed.

Finally, E is perfect. Let $p \in E$, and write it out as $p = 0.p_1p_2\ldots p_n\ldots$ as before. (note that each p_k is either 4 or 7). Let $\varepsilon > 0$, and let α be the position of the first significant digit of ε. (Thus if $\varepsilon = 0.0000001$ then $\alpha = 7$). Note that $\frac{1}{10^\alpha} < \varepsilon$. Let

$$q = 0.p_1p_2\ldots p_{\alpha-1}p_\alpha p'_{\alpha+1}p_{\alpha+2}\ldots,$$

where

$$p'_{\alpha+1} = \begin{cases} 4 & \text{if } p_{\alpha+1} = 7 \\ 7 & \text{if } p_{\alpha+1} = 4. \end{cases}$$

Since p and q only differ in the $\alpha + 1$th place, $|p-q| < \frac{1}{10^\alpha} < \varepsilon$. Clearly $p \neq q$ as well, so q is a point in E within ε of p other than p itself. Thus every point in E is a limit point of E, so E is perfect.

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18. *Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?*

There is, but this problem cannot be solved without employing Problem 28, which says that any closed set in a separable space such as $\mathbb{R}$ is the union of a perfect set P and some countable set, let's call it E.

This problem therefore boils down to finding an uncountable closed set that contains no rational numbers. Let $\{q\}_{n=1}^{\infty}$

be a sequence consisting of every rational number in $[0,10]$ (we can do this because the rationals are countable). Let

$$Q = \bigcup_{n=1}^{\infty} B\left(q, 2^{-n}\right);$$

that is, the infinite union of the ball of radius $\frac{1}{2}$ around q_1, the ball of radius $\frac{1}{4}$ around q_2, and so on. This is an infinite union of open sets, so it is itself open. Let $R = [0,10] - Q$. Several facts regarding R are now apparent:

- Since

  $$\begin{aligned} R &= [0,10] \cap \left(\bigcap_{n=1}^{q} B\left(q, 2^{-n}\right) \right) \\ &= [0,10] \cap \left[\bigcup_{n=1}^{q} \left(B\left(q, 2^{-n}\right)\right)^{c} \right], \end{aligned}$$

  R is the union of closed sets; R is therefore closed.

- R is bounded by $[0,10]$ so it is compact.

- Q clearly contains every rational number in $[0,10]$, so $\mathbb{R} - S$ contains none.

- I claim that R is nonempty. To see this, note that the total length of Q is at most

  $$\begin{aligned} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots &= 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= 1 + \frac{1}{1 - \frac{1}{2}} \\ &= 3. \end{aligned}$$

  Note that the 1 in front comes from the fact that the diameter of each ball is twice the radius. Note further that this is a *maximum* length, if every interval in Q was laid out end-to-end; they will likely overlap (and may extend beyond $[0,10]$) so the total length is likely smaller. That said, the remainder of R must therefore be uncountable and hence nonempty.

To summarize: R is a closed subset of $\mathbb{R}$ (a separable metric space) which contains no rational number. It is therefore the union of a perfect set P and an at most countable set E. Since R is uncountable, $P = R - E$ cannot be empty (and in fact is uncountable). Thus P is a nonempty perfect set containing no rational number.

19.

(a) *If A and B are disjoint closed sets in some metric space X, prove that they are separated.*

Since A and B are closed sets, $A = \overline{A}$ and $B = \overline{B}$. Thus $A \cap \overline{B} = A \cap B = \varnothing$ and $\overline{A} \cap B = A \cap B = \varnothing$.

(b) *Prove the same for disjoint open sets.*

Suppose $A \cap \overline{B}$ is not empty. Then there is p such that $p \in A$ and $p \in \overline{B}$. Since A is open, there exists a neighborhood $N_r(p) \subseteq A$ for some $r > 0$. Since $p \in \overline{B}$, it's in either B or B'. If it's in B then $p \in A \cap B$, a contradiction because A and B are disjoint. If $p \in B'$ then p is a limit point of B. This means that any neighborhood around p, and in particular $N_r(p)$ described above, has a point of B in it. But this neighborhood is a subset of A and they're supposed to be disjoint. This is another contradiction. Thus $A \cap \overline{B}$ must be empty.

(c) *Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.*

Suppose $A \cap \overline{B}$ is not empty. Then there exists x in both A and $\overline{B}$. Since $x \in A$, $d(p,x) < \delta$. Also, $x \in \overline{B} = B \cup B'$ means that $x \in B'$ (because if it were in B then $d(p,x) > \delta$). Thus x must be some limit point of B.

Let $r = \frac{\delta - d(x,p)}{2}$. Then $N_r(x)$ contains some $y \in B$. Clearly, we have $d(y,p) > \delta$. However,

$$\begin{aligned} d(y,p) &\leq d(y,x) + d(x,p) \\ &< r + d(x,p) \\ &= \frac{\delta - d(x,p)}{2} + d(x,p) \\ &= \frac{\delta + d(x,p)}{2} \\ &< \frac{\delta + \delta}{2} = \delta, \end{aligned}$$

a contradiction. Therefore $A \cap \overline{B}$ must be empty. A similar argument can be used to show that $\overline{A} \cap B$ must also be empty, so A and B are separated.

(d) *Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).*

This will be easier now that we've seen it on the test.

Let p, q be two distinct points in a connected metric space X. Then $0 < d(p,q)$. Let $r \in (0, d(p,q))$, and $S_r = \{x \in X : d(x,p) = r\}$.

Suppose S_r is empty. Let $U = B_r(p)$ and $V = \{x : d(x,p) > r\}$. Both of these sets are open, nonempty because $p \in U$ and $q \in V$, and disjoint. This means U and V form a separation of X, contradicting the supposition that X is connected. Thus S_r must be nonempty.

If we let r vary, we can create $f : (0, d(p,q)) \to X$ where $f(r)$ is a fixed point in S_r. This function is one-to-one, because $r_1 \neq r_2 \Rightarrow f(r_1) \neq f(r_2)$ (we can't have a point that's two different distances away from p.) Thus X must be uncountable because $(0, d(p,q))$ is uncountable.

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20. *Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)*

Suppose A and B are connected. Then without loss of generality suppose further that $A \cap \overline{B} \neq \varnothing$. Then there exists $x \in A \cap \overline{B}$, so $x \in A$ and $x \in \overline{B}$. But $A \subseteq \overline{A}$ and $\overline{B} = \overline{\overline{B}}$,so x is in both of them; that is, $x \in \overline{A} \cap \overline{\overline{B}}$, or in other words the closure of $\overline{A}$ and $\overline{B}$ is connected.

The interiors however are not necessarily connected. Consider in $\mathbb{R}$ the sets $A = [0,1]$ and $B = [1,2]$. These are connected because $1 \in A \cap \overline{B}$ as well as $1 \in \overline{A} \cap B$. However, $\overline{A} = (0,1)$ and $\overline{B} = (1,2)$, which are disjoint.

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21. *Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define*

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.

We need to show that $A_0 \cap \overline{B_0} = \varnothing$. Suppose not; that is, suppose there is $x \in A_0 \cap \overline{B_0}$. Then $x \in A_0$, and either $x \in B_0$ or x is a limit point of B_0. If $x \in B_0$, then $x \in A_0 \cap B_0$ so $\mathbf{p}(x) \in A \cap B$, which contradicts the presumption that A and B are separated. Thus x must be a limit point of B_0.

Let $\varepsilon > 0$ and consider

$$N = B\left(x; \frac{\varepsilon}{|\mathbf{b}-\mathbf{a}|}\right),$$

where $|\mathbf{b}-\mathbf{a}|$ is the distance between those two points in $\mathbb{R}^k$. Since x is a limit point of B_0, there must exist $y \neq x$ such that $y \in N$. Thus

$$d(x,y) < \frac{\varepsilon}{|\mathbf{b}-\mathbf{a}|}.$$

Furthermore, $y \in B_0$ so $y \in B$. With some manipulation we now have

$$\begin{aligned}
|\mathbf{p}(x) - \mathbf{p}(y)| &= |(1-x)\mathbf{a} + x\mathbf{b} - [(1-y)\mathbf{a} + y\mathbf{b}]| \\
&= |\mathbf{a} - \mathbf{a}x + x\mathbf{b} - \mathbf{a} + y\mathbf{a} - y\mathbf{b}| \\
&= |x(\mathbf{b}-\mathbf{a}) - y(\mathbf{b}-\mathbf{a})| \\
&= |(x-y)(\mathbf{b}-\mathbf{a})| \\
&< \left|\frac{\varepsilon}{|\mathbf{b}-\mathbf{a}|}(\mathbf{b}-\mathbf{a})\right| \\
&= \varepsilon.
\end{aligned}$$

This means that $\mathbf{p}(y)$ is within ε of $\mathbf{p}(x)$, which means $\mathbf{p}(x)$ is a limit point of B

This means that $\mathbf{p}(x) \in A \cap \overline{B}$ is a contradiction, since A and B are separated. Thus $A_0 \cap \overline{B_0} = \varnothing$, and by a similar argument $\overline{A_0} \cap B_0$ is empty, so A_0 and B_0 are separated.

(b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

Suppose not; that is, suppose $t_0 \in (0,1) \Rightarrow \mathbf{p}(t_0) \in A \cup B$. If $t \in (0,1)$ then since A and B are separated, $\mathbf{p}(t_0)$ can be in neither A nor B.. Also, $\mathbf{p}(t_0) \in A \cup B$ for all $t \in (0,1)$. Therefore $(0,1) = A_0 \cup B_0$. This is a contradiction since $(0,1)$ is a connected set.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

Let S be a convex subset of $\mathbb{R}^k$. If S is not connected, then S is the union of two nonempty separated sets A and B. By part (b) above there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$. But S is convex, so $\mathbf{p}(t_0) \in A \cup B$, which is clearly a contradiction. Thus S must be connected.

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27. *Define a point p in a metric space X to be a* condensation point *of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.*

Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Perfect is fairly straightforward. Suppose $p \in P$ is an isolated point. Then there is a neighborhood N of p such that $N \cap E$ is empty. This means that p is not a condensation point of E, which contradicts the definition of P. Thus P is a closed set which has no isolated points, which means it is perfect.

$\Rightarrow$: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and we will show that $P = W^c$. Suppose $x \in P$. Then x is a condensation point of E. If $x \in V_n$ for some n, then $E \cap V_n$ is uncountable because V_n is open. Therefore $x \in W^c$, because if $x \in W$ then there exists V_n such that $x \in V_n$ and $E \cap V_n$ is uncountable, a contradiction. Therefore $P \subset W^c$.

$\Leftarrow$: Suppose $x \in W^c$. Then $x \notin V_n$ if $E \cap V_n$ is countable. Let $N(x)$ be a neighborhood of x. Let $x \in V_n \subset N(x)$. Then $E \cap V_n$ is uncountable, which means $E \cap N(x)$ is also countable. Thus x is a condensation point of E. Therefore $W^c \subseteq P$.

We have subsets going both ways, so $P = W^c$.

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28. *Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable (Corollary: Every countable closed set in $\mathbb{R}^k$ has isolated points.) Hint: Use Exercise 27.*

Let X be a separable metric space, S be closed and let P be the set of condensation points of S. Since S is closed it contains all its limit points, and all condensation points are limit points. By above P is perfect and $S - P$ is countable and we're done.

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29. *Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. Hint: Use Exercise 22.*

This was done in class. Let G be an open set in $\mathbb{R}$. For each $x \in G$, define I_x as the union of all I such that I is an open interval, $x \in I$ and $I \subseteq G$. (This is called the *maximum interval* containing x.)

We need to show that I_x is an open interval. Because each I is open, I_x is open. This means I_x cannot be the isolated point $\{I_x\}$; it must contain at least two intervals. Since I_x is a family of connected sets such that $\{x\} \subseteq A_i \cap A_j$ for all $i, j \in I_x$, I_x is connected by practice problem 3, and so I_x is an interval.

Let $x, x' \in G$ such that $x \neq x'$. I claim then that either $I_x = I'_x$ or $I_x \cap I_{x'} = \varnothing$. To show this, suppose $I_x \cap I_{x'} \neq \varnothing$. Then $I_x \cup I_{x'}$ is an open interval that contains x and x'. Therefore $I_x \cup I_{x'} \subseteq I_x$ (because $I_x \cup I_{x'}$ participates in the union that defines I_x.) Therefore $I_x \subseteq I_x \cup I_{x'}$, and then applying the same argument to $I_{x'}$ solidifies the equality.

It is now clear that

$$G = \bigcup_{x \in G} I_x = \bigcup_{I_x \text{ disjoint}} I_x.$$

Therefore G is a countable union of disjoint open intervals.

30. *Imitate the proof of Theorem 2.43 to obtain the following result:*

- *If $\mathbb{R}^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.*
  Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \ldots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

Let G_n be dense open subsets of $\mathbb{R}^k$ for $n = 1, 2, 3, \ldots$. We need to show that $\bigcap_{n=1}^\infty G_n$ meets any nonempty subset of $\mathbb{R}^k$.

Let G_0 be a nonempty open subset of $\mathbb{R}^k$. Since G_1 is dense and G_0 is nonempty, $G_0 \cap G_1 \neq \varnothing$. Suppose $x_1 \in G_0 \cap G_1$. Since G_0 and G_1 are open, $G_0 \cap G_1$ is also open. Thus there is a neighborhood V_1 about x_1 such that $\overline{V_1} \subseteq G_0 \cap G_1$[1].

Now let's look at G_2. Again, G_2 is a dense open set and V_1 is a nonempty open set so $V_1 \cap G_2 \neq \varnothing$. Thus there is a nomepty open set V_2 such that $\overline{V_2} \subseteq V_1 \cap G_2$.

Suppose we have done this n times. Then we have n nonempty open sets $V_1, V_2, \ldots, V_n$ such that $\overline{V_1} \subseteq G_0 \cap G_1$ and $\overline{V_{i+1}} \subseteq V_i \cap G_{n+1}$ for all $i = 1, 2, \ldots, n-1$. Since G_{n+1} is a dense open set and V_n is a nonempty open set, $V_n \cap G_{n+1}$ is also a nonempty open set. Thus we have by induction a sequence of nested open sets $V_1, V_2, \ldots, V_n$ such that $\overline{V}_1 \subseteq G_0 \cap G_1$ and $\overline{V_{i+1}} \subseteq V_i \cap G_{i+1}$ for all $n \in \mathbb{N}$.

Since $\overline{V_1}$ is bounded (because V_1 a neighborhood so we're dealing with a closed ball around x_1) and $\overline{V_n} \subseteq \overline{V_{n-1}}$ for

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[1] Note that technically there is a V_1 such that $V_1 \subseteq G_0 \cap G_1$; however since we can make the diameter of V_1 as small as we want we can make V_1 closed if we wish.

$n = 2, 3, 4, \ldots$, we have by Theorem 2.39 that the intersection of the whole mess,

$$\bigcap_{n=1}^{\infty} \overline{V_n}$$

is not empty. Since $\overline{V_1} \subseteq G_0 \cap G_1$ and $\overline{V_{n+1}} \subseteq G_{n+1}$,

$$G_0 \cap \bigcap_{n=1}^{\infty} G_n$$

is not empty, and we are done.