

Homework, Vol. 1

Rudin

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Suppose $r+x$ is rational. Then $r+x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. But then $x = \frac{p}{q} - r = \frac{p-rq}{q}$, implying that x is rational which is a contradiction. Similarly, assuming rx is rational leads to $x = \frac{p}{qr}$ for some $p, q \in \mathbb{Z}$ and thus rational.

2. Prove that there is no rational number whose square is 12.

Suppose there exist $p, q \in \mathbb{Z}, q \neq 0$ such that they are relatively prime and $\left(\frac{p}{q}\right)^2 = 12$. Then $p^2 = 12q^2 = 3(2q)^2$. This means that $3 \mid p^2$ and $3 \mid p$. Thus $9 \mid p^2$ and $p^2 = 9k$ for some $k \in \mathbb{Z}$. $9k = 3(2q)^2$ implies that $9 \mid 3(2q)^2$, or $3 \mid (2q)^2$. This means that 3 divides either 2^2 or q^2 . We know $3 \nmid 4$, so it must divide q^2 . But this means that 3 divides both q and p , contradicting the assumption that they are relatively prime. Thus such p and q cannot exist, and so 12 cannot be the square of a rational number.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Since E is nonempty there's something in it; call it x . Since α is a lower bound of E it's at least as low as every thing in it; in particular $\alpha \leq x$. Similarly, β being an upper bound of E implies $x \leq \beta$, and a quick application of the transitive property yields $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Since A is bounded below there exists by Theorem 1.11 $\inf A$, call it α . Then for all $x \in A$ we have $\alpha < x$ and consequently $-\alpha > -x$. But this means that $-\alpha$ is an upper bound of $-A$.

Now, suppose there is another upper bound of $-A$, call it $-\alpha'$, such that $-\alpha' \leq -\alpha$. Then $-x \leq -\alpha'$ for all $-x \in -A$ and consequently $x \geq \alpha'$ for all $x \in A$. Furthermore, $-\alpha' \leq -\alpha$ implies $\alpha' \geq \alpha$. But $\alpha = \inf A$, so $\alpha' = \alpha$ and $-\alpha' = -\alpha$. Thus $-\alpha = \sup(-A)$, or $\alpha = -\sup(-A)$.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.

Let \mathbb{C}' be the set of complex numbers with some ordering applied, and suppose it is an ordered field. By 1.18d, $i^2 = -1 > 0$ and by 1.18a, $-(-1) = 1 < 0$. But this contradicts the second part of 1.18d (specifically, that the multiplicative identity is greater than the additive identity). Thus \mathbb{C}' cannot be an ordered field.

9. Suppose $z = a + bi, w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of relation is called a dictionary order, or lexicographic order, for obvious reasons.) Does this ordered set have the least upper bound property?

Let $S = \{a + bi \mid a < 0\}$. This is bounded by, say, $1 + 0i$. Let $\alpha + \beta i$ be another upper bound of S . If $\alpha > 0$, then $\alpha > \frac{\alpha}{2} > 0$ and $\frac{\alpha}{2}$ is another, smaller upper bound of S . If $\alpha = 0$, then $\alpha + (\beta - 1)i$ is also another, smaller boundary of S . In either case, we can always find a smaller upper bound of S , so S has no least upper bound. Thus the Least Upper Bound Property does not hold.

13. If x, y are real, prove that

$$||x| - |y|| \leq |x - y|.$$

We'll do two cases.

(a) If $|x| \geq |y|$ then $0 \leq |x| - |y|$, so $|x| - |y| \leq |x - y|$ implies that $||x| - |y|| \leq |x - y| = |x - y|$.

(b) If $|y| > |x|$ then $0 \leq |y| - |x|$, so $|x - y| = |y - x| \geq |y| - |x| \geq 0$, so we have $||y| - |x|| \leq |y - x|$, or $||x| - |y|| \leq |x - y|$.

15. Under what conditions does equality hold in the Schwarz inequality?

Suppose $b_j = ca_j$ for each j and some $c \in \mathbb{C}$. Then if each $a_j = x_j + iy_j$ we have

$$\begin{aligned} \left| \sum_{j=1}^n a_j \overline{b_j} \right|^2 &= \left| \sum_{j=1}^n a_j \overline{ca_j} \right|^2 \\ &= |c|^2 \left| \sum_{j=1}^n a_j \overline{a_j} \right|^2 \\ &= |c|^2 \left| \sum_{j=1}^n (x_j^2 + y_j^2) \right|^2 \\ &= |c|^2 \left[\sum_{j=1}^n (x_j^2 + y_j^2) \right]^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 &= \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |ca_j|^2 \\ &= |c|^2 \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |a_j|^2 \\ &= |c|^2 \left[\sum_{j=1}^n |a_j|^2 \right]^2 \\ &= |c|^2 \left[\sum_{j=1}^n (x_j^2 + y_j^2) \right]^2 \\ &= \left| \sum_{j=1}^n a_j \overline{b_j} \right|^2. \end{aligned}$$

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7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \leq \frac{a+b}{2}$ holds for all $a, b \geq 0$.

We'll work left to right. The first inequality is derived as follows:

$$\begin{aligned} a &< b \\ a^2 &< ab \\ a &< \sqrt{ab}. \end{aligned}$$

For the next step, we can do this:

$$\begin{aligned} (a-b)^2 &> 0 \\ a^2 - 2ab + b^2 &> 0 \\ a^2 + 2ab + b^2 &> 4ab \\ (a+b)^2 &> (2\sqrt{ab})^2 \\ a+b &> 2\sqrt{ab} \\ \frac{a+b}{2} &> \sqrt{ab}. \end{aligned}$$

The last step is the easy one:

$$\begin{aligned} a &< b \\ \frac{a}{2} &< \frac{b}{2} \\ \frac{a}{2} + \frac{b}{2} &< \frac{b}{2} + \frac{b}{2} \\ \frac{a+b}{2} &< b. \end{aligned}$$

Finally, stringing them all together gives us the inequalities we want.

13. The maximum of two numbers x and y is denoted by $\max(x, y)$. Thus $\max(-1, 3) = \max(3, 3) = 3$ and $\max(-1, -4) = \max(-4, -1) = -1$. The minimum of x and y is denoted by $\min(x, y)$. Prove that

$$\max(x, y) = \frac{x + y + |y - x|}{2},$$

$$\min(x, y) = \frac{x + y - |y - x|}{2}.$$

Derive a formula for $\max(x, y, z)$ and $\min(x, y, z)$, using, for example,

$$\max(x, y, z) = \max(x, \max(y, z)).$$

Suppose $x \leq y$. Then $\max(x, y) = y$ and $\min(x, y) = x$; also $|y - x| = y - x$. Thus we have

$$\begin{aligned} \frac{x + y + |y - x|}{2} &= \frac{x + y + y - x}{2} \\ &= \frac{2y}{2} \\ &= y \\ &= \max(y, x) \end{aligned}$$

and

$$\begin{aligned} \frac{x + y - |y - x|}{2} &= \frac{x + y - y + x}{2} \\ &= \frac{2x}{2} \\ &= x \\ &= \min(y, x). \end{aligned}$$

To derive $\max(x, y, z)$ using $\max(x, \max(y, z))$, we simply use two applications of the max formula and simplify:

$$\begin{aligned} \max(x, y, z) &= \max(x, \max(y, z)) \\ &= \frac{x + \max(y, z) + |\max(y, z) - x|}{2} \\ &= \frac{x + \frac{y+z+|y-z|}{2} + \left| \frac{y+z+|y-z|}{2} - x \right|}{2} \\ &= \frac{x + \frac{1}{2}(y+z+|y-z|) + \frac{1}{2}|y+z-2x+|y-z||}{2} \\ &= \frac{1}{2}x + \frac{1}{4}(y+z+|y-z|) + \frac{1}{4}|y+z-2x+|y-z||. \end{aligned}$$

$\min(x, y, z)$ can be done in the same manner:

$$\begin{aligned} \min(x, y, z) &= \min(x, \min(y, z)) \\ &= \frac{x + \min(y, z) - |\min(y, z) - x|}{2} \\ &= \frac{x + \frac{y+z-|y-z|}{2} - \left| \frac{y+z-|y-z|}{2} - x \right|}{2} \\ &= \frac{x + \frac{1}{2}(y+z-|y-z|) - \frac{1}{2}|y+z-2x-|y-z||}{2} \\ &= \frac{1}{2}x + \frac{1}{4}(y+z-|y-z|) - \frac{1}{4}|y+z-2x-|y-z||. \end{aligned}$$

14.

- (a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geq 0$. Why is it then obvious for $a \leq 0$?)

All right, we'll start with the case $a \geq 0$. Then $-a \leq 0$, and so by the definition of absolute value $|a| = a$ and $|-a| = -(-a) = a = |a|$.

If $a \leq 0$ then $-a \geq 0$, so $|a| = -a$ and $|-a| = -a$, and so we're done.

- (b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

\Rightarrow : Suppose that $-b \leq a \leq b$. If $a \geq 0$ then $a = |a| \leq b$. If $a < 0$ then $|a| = -a \Rightarrow a = -|a| \leq b$, so $|a| \leq b$.

\Leftarrow : Suppose that $|a| \leq b$. Since $a \leq |a|$, we have $a \leq |a| \leq b$ and $a \leq b$ by transitivity. But we also have $-a \leq |-a|$ and since we showed that $|-a| = |a|$ in part (a) we have $-a \leq |a| \leq b$, leading to $-a \leq b$ and $-b \leq a$. Stringing these together gives $-b \leq a \leq b$.

- (c) Use this fact to give a new proof that $|a+b| \leq |a| + |b|$.

We'll do three cases.

Since we know $a \leq |a| \Rightarrow -|a| \leq a \leq |a|$ and $b \leq |b| \Rightarrow -|b| \leq b \leq |b|$, we can write $-|a| - |b| \leq a + b \leq |a| + |b|$, or $-(|a| + |b|) \leq a + b \leq |a| + |b|$. Then the above property applies: $|a+b| \leq |a| + |b|$.

20. Prove that if

$$|x - x_0| < \frac{\epsilon}{2} \text{ and } |y - y_0| < \frac{\epsilon}{2},$$

then

$$|(x + y) - (x_0 + y_0)| < \epsilon,$$

$$|(x - y) - (x_0 - y_0)| < \epsilon.$$

Both of these can be done with some rearrangement and the Triangle Inequality:

$$\begin{aligned} |(x + y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &\leq |x - x_0| + |y - y_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

$$\begin{aligned} |(x - y) - (x_0 - y_0)| &= |(x - x_0) + (-y + y_0)| \\ &\leq |x - x_0| + |-y + y_0| \\ &= |x - x_0| + |y - y_0| \\ &\leq \epsilon \text{ by above.} \end{aligned}$$

22. Prove that if $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right),$$

then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon.$$

First, let's look at why y cannot be zero. Suppose it is. Then $|y - y_0| = |y_0|$, which is supposed to be less than the smaller of $\frac{|y_0|}{2}$ and $\frac{\epsilon|y_0|^2}{2}$. Obviously $|y_0|$ can't be smaller than $\frac{|y_0|}{2}$.

We can assume $y \neq 0$ because if it was then $|y_0| < \frac{|y_0|}{2}$

- First, $|y_0| < \frac{\epsilon|y_0|^2}{2}$. All these terms are in \mathbb{R}^+ , so after some manipulation we come to $\frac{2}{\epsilon} < |y_0|$
- Second, $\frac{\epsilon|y_0|^2}{2} < \frac{|y_0|}{2}$, which leads to $|y_0| < \frac{1}{\epsilon}$

Putting these together leads to $\frac{2}{\epsilon} < \frac{1}{\epsilon}$, which is impossible if $\epsilon \in \mathbb{R}^+$. This contradiction thus implies that $y \neq 0$. Whew!

Let $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right)$. Then $|y - y_0| < \frac{|y_0|}{2}$ and $|y - y_0| < \frac{\epsilon|y_0|^2}{2}$. Since $|y_0| - |y| < |y - y_0|$ we have $|y_0| - |y| < \frac{|y_0|}{2}$ by transitivity, leading to $\frac{|y_0|}{2} < |y|$ and $\frac{1}{|y|} < \frac{2}{|y_0|}$. Now we can write

$$\begin{aligned} \left|\frac{1}{y} - \frac{1}{y_0}\right| &= \frac{|y - y_0|}{|y| |y_0|} = \frac{1}{|y|} \cdot \frac{1}{|y_0|} \cdot |y - y_0| < \frac{2}{|y_0|} \cdot \frac{1}{|y_0|} \cdot |y - y_0| \\ &< \frac{2}{|y_0|^2} \cdot \frac{\epsilon|y_0|^2}{2} \\ &< \epsilon. \end{aligned}$$

23. Replace the question marks in the following statement by expressions involving ϵ , x_0 and y_0 so that the conclusion will be true:

If $y_0 \neq 0$ and

$$|y - y_0| < ? \text{ and } |x - x_0| < ?$$

then $y \neq 0$ and

$$\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \epsilon.$$

This problem is trivial in the sense that its solution follows from Problems 21 and 22 with almost no work at all (notice that $\frac{x}{y} = x \cdot \frac{1}{y}$). The crucial point is not to become confused; decide which of the two problems should be used first, and don't panic if your answer looks unlikely.

If we substitute $\frac{1}{y}$ for y and $\frac{1}{y_0}$ for y_0 then Problem 21 states that if

$$|x - x_0| < \min\left(\frac{1}{2} \cdot \frac{\epsilon |y_0|}{1 + |y_0|}, 1\right) \text{ and } \left| \frac{1}{y} - \frac{1}{y_0} \right| < \frac{\epsilon}{2|x_0| + 1}$$

then $\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \epsilon$. Now we need to figure out what $\left| \frac{1}{y} - \frac{1}{y_0} \right|$ is. Since $2|x_0| + 1 \geq 1$ for any x_0 we know that

$$\left| \frac{1}{y} - \frac{1}{y_0} \right| < \frac{\epsilon}{2|x_0| + 1} \leq \epsilon.$$

But this means we can pull out what we learned from Exercise 22, so we have: If

$$|x - x_0| < \min\left(\frac{1}{2} \cdot \frac{\epsilon |y_0|}{1 + |y_0|}, 1\right) \text{ and } |y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon |y_0|^2}{2}\right),$$

then $y \neq 0$ and

$$\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \epsilon.$$

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2. Find a formula for

(a) $\sum_{i=1}^n (2i - 1) = 1 + 3 + 5 + \dots + (2n - 1)$.

I suspect that this adds up to n^2 . I'll work by induction.

- The base case is easy as usual: 1^2 is indeed 1.
- Suppose $1 + 3 + \dots + (2n - 1) = n^2$. Then $1 + 3 + \dots + (2n - 1) + [2(n + 1) - 1] = n^2 + (2n + 1) = (n + 1)^2$.

(b) $\sum_{i=1}^n (2i-1)^2 = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2$.

Note that if we add all the even numbers to the original sum, we get the sum of the squares of all the numbers from 1 to $2n$. With this we can deduce that

$$\begin{aligned} [1^2 + 3^2 + 5^2 + \dots + (2n-1)^2] + [2^2 + 4^2 + \dots + (2n)^2] &= \sum_{i=1}^{2n} n^2 \\ [1^2 + 3^2 + 5^2 + \dots + (2n-1)^2] + 4[1^2 + 2^2 + \dots + n^2] &= \sum_{i=1}^{2n} n^2 \\ [1^2 + 3^2 + 5^2 + \dots + (2n-1)^2] &= \sum_{i=1}^{2n} n^2 - 4 \sum_{i=1}^n n^2 \\ [1^2 + 3^2 + 5^2 + \dots + (2n-1)^2] &= \frac{2n(2n+1)(4n+1)}{6} - 4 \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(2n+1)(2n-1)}{3} \end{aligned}$$

Hint: what do these expressions have to do with $1 + 2 + 3 + \dots + 2n$ and $1^2 + 2^2 + 3^2 + \dots + (2n)^2$?

3. If $0 \leq k \leq n$, the “binomial coefficient” $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}, \text{ if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$

• (a) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(The proof does not require an induction argument.)

I'll work this backwards.

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)![n-(k-1)]!} + \frac{n!}{k!(n-k)!} \\ &= \frac{k \cdot n!}{(n-k+1)k!(n-k)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \left(\frac{k}{n+1-k} + 1 \right) \\ &= \frac{n!}{k!(n-k)!} \left(\frac{n+1}{n+1-k} \right) \\ &= \frac{(n+1)!}{k![(n+1)-k]!} \\ &= \binom{n+1}{k} \end{aligned}$$

- (d) Prove the “binomial theorem”: If a and b are any numbers and n is a natural number, then

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + b^n \\ &= \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j.\end{aligned}$$

The case for $n = 1$ isn't hard:

$$\begin{aligned}\sum_{j=0}^1 \binom{n}{j} a^{n-j} b^j &= \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 \\ &= a + b \\ &= (a+b)^1.\end{aligned}$$

Suppose

$$(a+b)^{n-1} = a^{n-1} + \binom{n-1}{1} a^{n-1-1}b + \binom{n-1}{2} a^{n-1-2}b^2 + \cdots + \binom{n-1}{n-1-1} ab^{n-1-1} + b^{n-1}.$$

Then multiplying both sides by $a+b$ yields

$$\begin{aligned}(a+b)^n &= (a+b) \left[a^{n-1} + \binom{n-1}{1} a^{n-1-1}b + \binom{n-1}{2} a^{n-1-2}b^2 + \cdots + \binom{n-1}{n-1-1} ab^{n-1-1} + b^{n-1} \right] \\ &= a^n + \binom{n-1}{1} a^{n-1}b + \binom{n-1}{2} a^{n-2}b^2 + \cdots + \binom{n-1}{n-2} a^2 b^{n-2} + ab^{n-1} \\ &\quad + a^{n-1}b + \binom{n-1}{1} a^{n-2}b^2 + \binom{n-1}{2} a^{n-3}b^3 + \cdots + \binom{n-1}{n-2} ab^{n-1} + b^n \\ &= a^n + [n-1+1] a^{n-1}b + \left[\binom{n-1}{2} + \binom{n-1}{1} \right] a^{n-2}b^2 + \cdots \\ &\quad + \left[\binom{n-1}{n-2} + \binom{n-1}{n-3} \right] a^2 b^{n-2} + (n-1+1) a^{n-1}b + b^n \\ &= a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a^{n-1}b + b^n \\ &= \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j.\end{aligned}$$

5.

(a) Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$.

The case for $n = 1$ is easy enough:

$$\begin{aligned} \frac{1 - r^2}{1 - r} &= \frac{(1 - r)(1 + r)}{1 - r} \\ &= 1 + r. \end{aligned}$$

Suppose that

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

Then

$$\begin{aligned} 1 + r + r^2 + \cdots + r^{n-1} + r^n &= \frac{1 - r^n}{1 - r} + r^n \\ &= \frac{1 - r^n + r^n - r^{n+1}}{1 - r} \\ &= \frac{1 - r^{n+1}}{1 - r}. \end{aligned}$$

(b) Derive this result by setting $S = 1 + r + \cdots + r^n$, multiplying this equation by r , and solving the two equations for S .

$$\begin{aligned} S &= 1 + r + \cdots + r^n \\ rS &= r + r^2 + \cdots + r^{n+1}. \end{aligned}$$

Subtracting the second equation from the first yields

$$\begin{aligned} S - rS &= 1 + r + \cdots + r^n - r - r^2 - \cdots - r^{n+1} \\ S(1 - r) &= 1 - r^{n+1} \\ S &= \frac{1 - r^{n+1}}{1 - r}. \end{aligned}$$

6. The formula for $1^2 + \dots + n^2$ may be derived as follows. We begin with the formula

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \dots, n$ and adding, we obtain

$$\begin{aligned} 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ &\vdots \end{aligned}$$

$$\frac{(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1}{(n+1)^3 - 1 = 3[1^2 + \dots + n^2] + 3[1 + \dots + n] + n.}$$

Thus we can find $\sum_{k=1}^n k^2$ if we already know $\sum_{k=1}^n k$ (which could have been found in a similar way). Use this method to find

- (a) $1^3 + \dots + n^3$.
We start with

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

for $k = 1, \dots, n$. From this we have

$$\begin{aligned} (n+1)^4 - 1 &= 4 \sum_{k=1}^n k^3 + 6 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k + n \\ \sum_{k=1}^n k^3 &= \frac{1}{4} \left[(n+1)^4 - 1 - 6 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k - n \right] \\ &= \frac{1}{4} \left[(n+1)^4 - 1 - 6 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} - n \right] \end{aligned}$$

which after a whole lot of algebra simplifies to $\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$.

- (b) $1^4 + \dots + n^4$.
Should be similar to the last one.

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

and so

$$\begin{aligned} (n+1)^5 - 1 &= 5 \sum_{k=1}^n k^4 + 10 \sum_{k=1}^n k^3 + 10 \sum_{k=1}^n k^2 + 5 \sum_{k=1}^n k + n \\ \sum_{k=1}^n k^4 &= \frac{1}{5} \left[(n+1)^5 - 1 - 10 \sum_{k=1}^n k^3 - 10 \sum_{k=1}^n k^2 - 5 \sum_{k=1}^n k - n \right] \\ &= \frac{1}{5} \left[(n+1)^5 - 1 - 10 \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \right) - 10 \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) - 5 \left(\frac{n^2}{2} + \frac{n}{2} \right) - n \right] \end{aligned}$$

and again after all the manipulation is done we get $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

Handout

3. Suppose that the number a has the property that for every natural number n , $a \leq \frac{1}{n}$. Prove that $a \leq 0$.

Suppose that $a > 0$. By the Archimedean Principle there exists $n \in \mathbb{N}$ such that $na > 1$. But this means that $a > \frac{1}{n}$. Thus we have the following: $a > 0$ implies that there exists $n \in \mathbb{N}$ such that $a > \frac{1}{n}$, which is the contraposition of what we wish to prove.

4. Given a real number a , define $S = \{x \mid x \in \mathbb{Q}, x < a\}$. Prove that $a = \sup S$.

We know that S is bounded above by a because that's how we defined it. Since S is a set of real numbers, we know that $\sup S$ exists as a real number. But what if some other real number, call it b , is the supremum of S ? We'll look at two cases.

- If $b > a$, then there exists $q \in S$ such that $a < q \leq b$. But this is impossible since a is an upper bound of S .
- If $b < a$ then let $b' = \frac{a+b}{2}$ (note that $b < b'$). Then there exists $q \in \mathbb{Q}$ such that $b' \leq q < a$. This again proves to be trouble because it means that $b < q$ and b is no longer an upper bound for S (b must be an upper bound of S if it is to be the supremum). Thus this case is also impossible.

Therefore by trichotomy we must have $b = a$, and so $a = \sup S$.

5. Prove that every bounded, nonempty set of natural numbers has a maximum.

First, note that since \mathbb{N} is bounded it has a finite number of elements. We'll proceed by induction. Let $S_n = \{a_1, a_2, \dots, a_n\}$ be a bounded set of natural numbers with n elements. Obviously, $S_1 = \{a_1\}$ has a maximum since a_1 is the only thing in it. Suppose $S_k = \{a_1, a_2, \dots, a_k\}$ has a maximum α , and consider $S_{k+1} = S_k \cup \{a_{k+1}\}$. If $\alpha \geq a_{k+1}$, then it is the maximum of S_{k+1} ; if not then a_{k+1} is the maximum. Either way S_{k+1} has a maximum, and so by induction the premise holds true.

6. Let A and B be two sets of real numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let C (E) be the set of all sums (products) of the form $x + y$ (xy), where $x \in A$ and $y \in B$. Is it necessarily true that $a + b = \sup C$ ($ab = \sup E$)? Prove or disprove.

- $a + b = \sup C$: Let $c = a + b$. We first need to show that c is an upper bound for C . Let $x \in C$. Then $x = a' + b'$, where $a' \in A$ and $b' \in B$. This implies that $a' \leq a$ and $b' \leq b$. But this means $a' + b' \leq a + b$, implying $x \leq c$ by transitivity.

We now know that c is an upper bound, but still need to show it is the *least* upper bound. Suppose it is not; that is, suppose $N = \sup \{C\}$ and that $c > N$. Then $N = c - \alpha$ for some positive α . Thus if we let $\varepsilon = \frac{\alpha}{4}$ there exists an element $a_1 \in A$ such that $a_1 \geq a - \varepsilon$. Similarly, $b_1 \geq b - \varepsilon$ for some $b_1 \in B$. This implies $a_1 + b_1 \geq a + b - 2\varepsilon = c - \frac{\alpha}{2}$. But $a + b \in C$, and $c - \frac{\alpha}{2} > c - \alpha = N$, and so we have found an element in C that is greater than N ; contradicting the assumption that N is an upper bound. Therefore the upper bound $N < c$ cannot exist, and so c must be the least upper bound.

- $ab = \sup E$: This turns out to be *not* true in general. Let $a = (-\infty, 0)$ and $b = \{-1\}$ then $E = (0, \infty)$, which clearly lacks a supremum.

7. (The monotone property for suprema) Suppose that $A \subseteq B$ are nonempty subsets of \mathbb{R} . If B has a supremum, prove that $\sup A$ exists and $\sup A \leq \sup B$. Formulate and prove a similar statement for infima.

Let $a \in A$. Then $a \in B$. Since B has a supremum, $a \leq \sup B$. Thus A is bounded above by $\sup B$, and since $A \subseteq \mathbb{R}$ we know that $\sup A$ exists.

Suppose $\sup B < \sup A$. Then if $\alpha = \frac{\sup A + \sup B}{2}$ there exists $a' \in A$ such that $a' \in [\alpha, \sup A]$. Since $\sup B < \alpha$ we have $\sup B < \alpha$. But this means that α is greater than every element in B and thus cannot be a member of B . This contradicts the fact that $A \subseteq B$; thus $\sup A$ cannot be greater than $\sup B$, and we are done.

The equivalent statement for infima is as follows: Suppose that $A \subseteq B$ are nonempty subsets of \mathbb{R} . If B has an infimum, then $\inf A$ exists and $\inf B \leq \inf A$.

Proof: Let $a \in A$. Then $a \in B$. Since B has an infimum, $a \geq \inf B$. Thus A is bounded below by $\inf B$, and since $A \subseteq \mathbb{R}$ we know that $\inf A$ exists.

Suppose $\inf B > \inf A$. Then if $\alpha = \frac{\inf B + \inf A}{2}$ there exists $a' \in A$ such that $a' \in [\inf A, \alpha]$. Since $\inf B > \alpha$ we have $\inf B > \alpha$. But this means that α is less than every element in B and thus cannot be a member of B . This contradicts the fact that $A \subseteq B$; thus $\inf A$ cannot be greater than $\inf B$, and we again are done. Note that this could also be proven with Exercise 5 from Rudin, but it's still a lot of typing.

8. Suppose $A, B \subseteq \mathbb{R}$ and $E = A \cup B$. If E has a supremum and both A and B are nonempty, prove that $\sup A$ and $\sup B$ both exist, and $\sup E$ is one of the numbers $\sup A$ and $\sup B$. What can you say about $\inf(A \cup B)$? $\sup(A \cap B)$? $\inf(A \cap B)$? (Assume $A \cap B \neq \emptyset$). Must support your assertion.

Let $M = \max\{\sup A, \sup B\}$, and assume without loss of generality that $\sup A \geq \sup B$. Clearly, M is an upper bound for E as M is an upper bound to both A and B and is thus greater than every element in both. Suppose M is not the least upper bound of E . Then the least upper bound is some N such that $N < M$. But then N is an upper bound for any subset of E ; in particular, N is an upper bound for A . But this is impossible, as M is the *least* upper bound for A and $N < M$. Thus we have a contradiction and conclude that M must be the least upper bound of E and is either $\sup A$ or $\sup B$.

As for the other three questions:

- $\inf(A \cup B)$: Let $m = \min\{\inf A, \inf B\}$, and assume without loss of generality that $\inf A \leq \inf B$. Clearly, m is a lower bound for $A \cup B$ as m is a lower bound to both A and B and is thus less than every element in both. Suppose $m \neq \inf(A \cup B)$. Then the greatest lower bound is some m' such that $m < m'$. But then m' is a lower bound for any subset of $A \cup B$; in particular, m' is a lower bound for A . But this is impossible, as m is the *greatest* lower bound for A and $m < m'$. Thus we have a contradiction and conclude that m must be the greatest lower bound of $\inf A \cup B$ and is either $\inf A$ or $\inf B$.
- $\sup(A \cap B)$: Let $M = \min\{\sup A, \sup B\}$ and suppose without loss of generality that $\sup A \leq \sup B$. If $x > M$, then $x \notin A$ so $x \notin A \cap B$; by contraposition this means that $x \in A \cap B \Rightarrow x \leq M$ and M is an upper bound for $A \cap B$. This means that the *least* upper bound of $A \cap B$ is less than or equal to $\min\{\sup A, \sup B\}$. Can we say more than that? Not really, apart from saying that it exists (because $A \cap B \neq \emptyset$). For example, suppose $A = [0, 100]$ and $B = [0, x] \cup \{10000\}$ for some $x \in (0, 100)$. Then $A \cap B = (0, 100)$ and the supremum should be whatever x is, but we can't say more than that. We can't even say whether or not $\sup(A \cap B)$ is an element of either A or B , let alone $A \cap B$ (suppose $A = B = (0, 1)$, for example).
- $\inf(A \cap B)$: One last time: let $m = \max\{\inf A, \inf B\}$ and suppose without loss of generality that $\sup A \geq \sup B$. If $x < m$, then $x \notin A$ so $x \notin A \cap B$; by contraposition this means that $x \in A \cap B \Rightarrow x \geq m$ and m is a lower bound for $A \cap B$. As before, we can't say anything stronger than this apart from its existence.