Jesse Gage

Math 230A

Dr. Zhou

October 9, 2006

Homework, Vol. 2

Handout

1. *If $\{1,2,\dots,n\} \sim \{1,2,\dots,m\}$, prove that $m=n$.*
From class, we know that $\{1,2,\dots,n\} \sim \{1,2,\dots,m\}$ if and only if the former has exactly m elements. But $\{1,2,\dots,n\} \sim \{1,2,\dots n\}$, so it also has exactly n elements. Thus $m=n$.

---

2. *Let S be a finite set containing of n elements and let $\mathcal{P}(S)$ be the collection of all subsets of S. Show that $\mathcal{P}(S)$ is a finite set and find the number of elements in $\mathcal{P}(S)$.*
(Note that I'm using $\mathcal{P}(S)$ instead of T to denote the power set of S.) We can show that $\mathcal{P}(S)$ is a finite set by simply calculating the number of elements in $\mathcal{P}(S)$. We'll work by induction.
Let $S_n = \{s_1, s_2, \dots, s_n\}$ denote a set with cardinality n. For $S_1 = \{s_1\}$, $\mathcal{P}(S) = \{\varnothing, S_1\}$ has 2^1 elements. Suppose $\mathcal{P}(S_{n-1})$ has 2^{n-1} elements. Then the power set of S_n will consist of all the elements of S_{n-1}, together with the set of subsets of S_{n-1} unioned with $\{s_n\}$. We can thus equate the cardinalities:

$$\begin{aligned} |\mathcal{P}(S_n)| &= |\mathcal{P}(S_{n-1})| + |\{A \mid A = \{s_n\} \cup B \forall B \subseteq S_{n-1}\}| \\ &= |\mathcal{P}(S_{n-1})| + |\mathcal{P}(S_{n-1})| \\ &= 2|\mathcal{P}(S_{n-1})| \\ &= 2 \cdot 2^{n-1} \\ &= 2^n. \end{aligned}$$

Thus the power set of S_n has 2^n elements, which is finite.

---

3. *Find a 1-1 function f from $\mathbb{N}$ onto S where S is the set of all odd numbers explicitly.*
Let $f : \mathbb{N} \to S$ be defined by

$$f(x) = \begin{cases} x & x \text{ is odd} \\ 1-x & x \text{ is even.} \end{cases}$$

(Note that $f(x)$ is never zero.) Suppose $f(a) = f(b)$. If they are positive, then we have $a=b$ directly. If they are negative, then we have $1-a = 1-b$, which also leads to $a=b$. thus f is one-to-one.
f is also onto: if $x \in S$ is positive then $f(x) = x$ and if it's negative then $f(1-x) = 1-(1-x) = x$.

---

4. *Let P_n be the set of all polynomials of degree n with integer coefficients. Prove that P_n is countable. (Hint: A proof by induction is one method of approach.)*
This might actually be easier if we don't use induction. Let $f : \mathbb{Z}^{n+1} \to P_n$ be defined by

$$f(a_0, a_1, \dots, a_n) = a_0 + a_1 x + \dots a_n x^n.$$

Let $\phi = p_0 + p_1 x + \dots + p_n x^n \in P_n$. Then $f(p_0, p_1, \dots p_n) = \phi$. Since we know that $\mathbb{Z}^{n+1}$ is countable (it is the finite product of countable sets) and we have mapped it onto P_n, it follows that P_n must be countable.

---

5. *Use the above result to prove that the set of all polynomials with integers is a countable set.*
Let P be the set of all polynomials. Then $P = \bigcup_{\alpha=1}^{\infty} P_\alpha$. But this is a countable union of countable sets, and so P is itself countable.

---

6. *For each $p \in P_n$ (P_n as defined in #4), define $B(p) = \{x : p(x) = 0\}$. Prove that $\bigcup_{p \in P_n} B(p)$ is countable.*

The Fundamental Theorem of Algebra guarantees that there are at most n unique real roots to any polynomial $p(x)$. Thus we can label the solutions $a_1, \ldots, a_n$, let's say in ascending order (these are solutions in $\mathbb{R}$ so we can do so). Since that cannot be larger than $\mathbb{N}$ we can easily modify the function from Problem 4 to suit our needs. Let $g_n : \mathbb{Z}^{n+1} \times \mathbb{N} \to B(p) \times \{1, 2, 3, \ldots, n\}$ be defined such that $f(c_0, c_1, c_2, \ldots, c_n, m)$ is the mth solution (remember we can order them because they're finite). To show that f is onto we let α be the kth solution to $c_0 + c_1 x + \cdots + c_n x^n = 0$. Then

$$f(c_0, c_1, \ldots, c_n, k) = \alpha.$$

Since $\mathbb{Z}^{n+1} \times \mathbb{N}$ is countable and maps onto $B(p)$ it must be at most countable. We also know that $B(p)$ is infinite (there are infinite possibilities for $x + c = 0$ for example) it must be strictly countable.

---

7. (Rudin problems are listed at the end.)

---

8. *Show that the following sets are countable:*

(a) *The set of circles in the complex plane having rational radii and centers with rational coordinates.*
Let's call the set C. This can be described as $\{(x, y, r) : x, y, r \in \mathbb{Q} \text{ and } x^2 + y^2 = r^2\}$. Let $f : \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \to C$ be defined as $f(x, y, z)_{\mathbb{Q}} = (x, y, z)_C$. There's no problem showing that this is onto: if $(x, y, z)_C$ is a circle in the complex plane having radius z and center (x, y)then $f(x, y, z)$ yields that circle. Thus we have mapped the countable set $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ onto C, so C must be at most countable. However, C is clearly infinite (there's an infinite number of circles at coordinates $(0, 0)$ for example) so C must actually be countable.

(b) *Any collection of disjoint intervals of positive length.*
Let S be the collection in question. We know that the empty set is countable (There's zero elements), so let's suppose there's at least one element (that is, interval) $S_0 \in S$. Note that within any positive length interval there exist at least one rational number, because $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $f : \mathbb{Q} \to S$ be defined as

$$f(x) = \begin{cases} A & x \in A \text{ for some } A \in S \\ S_0 & \text{otherwise} \end{cases}$$

We need to show that f is onto. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we know any interval $A \in S$ there is at least one rational number $x \in A$; thus $f(x) = A$. f is therefore onto, and since the countable set $\mathbb{Q}$ is mapped onto S we know that S is countable.

---

9. *Let f be a real-valued function defined for every x in the interval $0 \le x \le 1$. Suppose there is a positive number M having the following property: for every choice of a finite number of points $x_1, \ldots, x_n$ in the interval $0 \le x \le 1$, the sum $|f(x_1) + \cdots + f(x_n)| \le M$. Let S be the set of $0 \le x \le 1$ for which $f(x) \ne 0$. Prove that S is countable.*

I'm assuming countable here means *at most countable*. Let's look at S^+, the positive values of S first. For $n \in \mathbb{N}$, let $I_0 = (1, \infty)$. By the Archimedian Principle, there can be no more than M elements greater in I_0, because if there were more we could sum them up to get a value greater than M. Let $S_0^+ = \{x : f(x) > 1\}$. Now let $I_n = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ for $n = 1, 2, 3 \ldots$, and let $S_n^+ = \{x : f(x) \in I_n\} \subseteq S^+$. Again, by Archimedes there are at most $\frac{M}{2^k}$ elements of S_n^+ in any interval I_k. Noting that $\bigcup_{k=0}^{\infty} I_k = (0, 1]$ and $S^+ = \bigcup_{k=0}^{\infty} S_k$, we have expressed S^+ as a countable collection of finite sets, and is therefore itself countable. Similarly, S^-, the negative values of S is also countable, and so $S = S^+ \cup S^-$ is countable.

10. *The purpose of this problem is to show that the open interval $(0,1)$ is equivalent to the closed interval $[0,1]$. In the process we will discover that both intervals are equivalent to $[0,1)$ and $(0,1]$. It is then easy to generalize to any interval $[a,b]$ with $a<b$. In each case, you need to find an explicit equivalence relation.*

Define $f:(0,1)\to\mathbb{R}$ as follows. For $n\in\mathbb{N}$, $n\geq 2$, $f\left(\frac{1}{n}\right)=\frac{1}{n-1}$ and for all other $x\in(0,1)$, $f(x)=x$.

(a) *Prove that f is a 1-1 function from $(0,1)$ into $(0,1]$.*
Suppose $a\neq b$. If a is of the form $\frac{1}{n}$ for $n\geq 2$ and b is not then $f(a)$ will be of the form $\frac{1}{n}$ and $f(b)$ won't be, so $f(b)\neq f(a)$.
If $a=\frac{1}{n}$ and $b=\frac{1}{m}$ with $m,n\geq 2$ and $m\neq n$ then $f(a)=\frac{1}{n-1}$ and $f(b)=\frac{1}{m-1}$. Suppose they are equal. Then we have $\frac{1}{n-1}=\frac{1}{m-1}\to m=n$, contradicting our assumption that $m=n$.
Otherwise, $f(a)=a$ and $f(b)=b$, so $f(a)\neq f(b)$.
We have thus shown that $f(a)$ cannot equal $f(b)$ if $a\neq b$, which is the contraposition of what we wish to prove.

(b) Prove that f is a function from $(0,1)$ onto $(0,1]$.
Let $y\in(0,1]$. If $y=\frac{1}{n}$ for some $n\geq 1$, then $f\left(\frac{1}{n+1}\right)=\frac{1}{n}=y$. If not, then $f(y)=y$. In either case we have something from $(0,1)$ being mapped to $(0,1]$ so f is onto.

(c) *Find a 1-1 function from $[0,1)$ onto $[0,1]$.*
Define $g:[0,1)\to[0,1]$ by

$$g(x)=\begin{cases}0 & \text{if } x=0\\ f(x) & \text{otherwise,}\end{cases}$$

Where $f(x)$ is as described above. We need to show that g is one-to-one and onto.

- One-to-one: Suppose $g(a)=g(b)$. If $g(a)=0$ then $a=0$ and $b=0$ because 0 is the only value for which $g(x)=0$, so $a=b$. If not, then $a=b$ for the reason outlined in part (a).
- Onto: If $c=0$ then $g(0)=c$; otherwise we can refer to the argument in part (b).

(d) Prove that $[0,1)$ is equivalent to $(0,1]$.
Define $h:[0,1)\to(0,1]$ by

$$h(x)=1-x$$

We need to show that h is 1-1 and onto.

- 1-1: Suppose $f(a)=f(b)$. Then since $h(x)=1-x\to 1-h(x)=x$, we have $1-f(a)=a$ and $1-f(a)=b$, leading to $a=b$.
- Onto: let $c\in(0,1]$. Then $f(1-c)=1-(1-c)=c$.

(e) Prove that $(0,1)$ is equivalent to $[0,1]$.
Combining all these, we have $(0,1)\sim(0,1]\sim[0,1)\sim[0,1]$ and $(0,1)\sim[0,1]$ by transitivity. If you want an explicit function then we can compose $f\circ h\circ g$ to get what we want.

---

11. *Let a,b,c and d be any real numbers such that $a<b$ and $c<d$. Prove that $[a,b]$ is equivalent to $[c,d]$. Find an equivalence function explicitly.*

A straight line in $\mathbb{R}^2$ should do nicely. The line $f:\mathbb{R}\to\mathbb{R}$ connecting (a,c) and (b,d) is $f(x)=\left(\frac{d-c}{b-a}\right)(x-a)+c$. Note that $f(c)=a$ and $f(b)=d$. Since $c\neq d$ and $a\neq b$ we know the line is neither vertical nor horizontal and thus one-to-one and onto, and consequently $[a,b]\sim[c,d]$.

12. *If A is a countable set and B an uncountable set, prove that $B \backslash A \sim B$.*
(Note that the problem is trivial if A and B are disjoint.) A quick lemma before we get started.

Lemma. $B \backslash A$ is uncountable.

Proof. Suppose $B \backslash A$ is countable. Then $B = (B \backslash A) \cup A$ is the union of two at most countable sets, so B is countable, which is a contradiction of our assumption.

Now for the real proof. From class, there exists a countable set $C \subseteq B \backslash A$, which of course is also in B. Let's label the elements of C as $\{c_1, c_2, \ldots\}$ and the elements of $A \cap B$ as $\{a_1, a_2, a_3, \ldots\}$ Now define $f : B \backslash A \to B$ by

$$f(x) = \begin{cases} a_{i/2} & \text{if } x = c_i \text{ where } i \text{ is even} \\ c_{(i+1)/2} & \text{if } x = c_i \text{ where } i \text{ is odd} \\ x & x \notin C. \end{cases}$$

We need to show that f is one-to-one and onto.
One-to-one: Suppose $f(x) = f(y)$.
Suppose $x \neq y$. Note that $f(\{c_i : i \text{ even}\})$, $f(\{c_i : i \text{ odd}\})$ and $f(B \backslash (A \cup C))$ map to three disjoint areas of B and form a partition. Thus if x and y are not in the same set then $f(x) = f(y)$ is impossible. We thus just need to check these three cases.

(a) If $a = c_k$ and $b = c_j$ where k and j are even but $k \neq j$, then $f(a) = a_{k/2}$ and $f(b) = a_{j/2}$, which are two different elements of A, so $f(a) \neq f(b)$.

(b) Similarly, if k and j are odd then $f(a)$ and $f(b)$ map two two different elements of C so $f(a) \neq f(b)$.

(c) Finally, if a and b are not in C then we have $a = f(a)$ and $b = f(b)$ so $f(a) \neq f(b)$ directly.

Thus we have one-to-one by contraposition.
Onto: if $y \in B$ we need $x \in B \backslash A$ such that $f(x) = y$. There's three cases:

(a) If $y = a_k$ for some $a_k \in A \cap B$ then $f(c_{2k}) = a_k$.

(b) If $y = c_k$ for some $c_k \in C$ then $f(c_{2k-1}) = c_k$.

(c) If $y \notin (A \cap B) \cup C$ then $f(y) = y$.

To summarize: we have shown that f is one-to-one and onto, and so $B \backslash A \sim B$.

---

13. *Prove Dedekind's theorem: A set S is infinite iff there is a proper subset A of S such that A is equivalent to S.*
$\Rightarrow$: We'll break this into two cases, one where S is countable and one where it's not. The case where S is uncountable stems from Problem 12. Since S is uncountable there exists a countable set $A \subseteq S$; $S \backslash A \subset S$ but $S \backslash A \sim S$.
If S is countable, then $S = \{s_1, s_2, s_3, \ldots\}$. Define $f : S \backslash \{s_1\} \to S$ as $f(s_n) = s_{n-1}$. Since f is quite clearly one-to-one and onto, we have $S \backslash \{s_1\} \subset S$ but $S \backslash \{s_1\} \sim S$.
$\Leftarrow$: This is much easier. Suppose S is finite and let $A \subset S$. Then $|S| > |A|$, so $S \not\sim A$ by Problem 1, giving us the contraposition of what we wish to prove.

---

14. *Let $\mathbb{R}$ denote the set of real numbers and let S denote the set of all real-valued functions whose domain is $\mathbb{R}$. Show that S and $\mathbb{R}$ are not equivalent. Hint: Assume $S \sim \mathbb{R}$ and let f be a one-to-one function such that $f(\mathbb{R}) = S$. If $a \in \mathbb{R}$, let $g_a = f(a)$ be the real-valued function in S which corresponds to the real number a. Now define h by the equation $1 + g_x(x)$ if $x \in \mathbb{R}$, and show that $h \notin S$.*
Define f, g_a and h as above. Suppose $h \in S$. Then there exists $a \in \mathbb{R}$ such that $f(a) \equiv h$; that is to say, $g_a \equiv 1 + g_x(x)$ and $g_a(x) = 1 + g_x(x)$ for **all** $x \in \mathbb{R}$, and in particular $x = a$. But then we have $g_a(a) = 1 + g_a(a) \to 0 = 1$ which is obviously a contradiction. Thus h cannot be in S, which itself is a contradiction of the assumption that S is the set of all real-valued functions. Therefore we must conclude that $\mathbb{R}$ and S are not equivalent.

15. *For a set A, let $\mathcal{P}(A)$ be the set of all subsets of A. Prove that A is not equivalent to $\mathcal{P}(A)$. Hint: Suppose $f : A \to \mathcal{P}(A)$ and define $E = \{x : x \in A$ and $x \notin f(x)\}$. Show $E \notin img(F)$, where $img(F)$ is defined as*

$$img(F) = \{B : B \subseteq A \text{ and there is } x \in A \text{ such that } F(x) = B\}.$$

How is this problem related to #14?
(Note that I'm using F instead of f to remind myself that it's a function that gives a set instead of a simple element.) Suppose $E \in \text{img}(F)$. Then there exists $x_0 \in A$ such that $F(x_0) = E$. We know that x_0 is either in $F(x_0)$ or it's not in $F(x_0)$.

- Suppose $x_0 \in F(x_0) = E$ then $x_0 \in \{x : x \in A$ and $x \notin F(x)\}$. But this means that $x_0 \notin F(x_0)$, which is a contradiction.
- Now suppose $x_0 \notin F(x_0)$ then $x_0 \in E$. But this implies $x_0 \in F(x_0)$ which again is a contradiction.

Either way we reach a contradiction, and so we can only conclude that $E \notin \text{img}(F)$
This is related to Problem 14 because the range of all real valued functions of $\mathbb{R}$ is $\mathcal{P}(\mathbb{R}) \setminus \varnothing$.

Rudin

Page 43

2. *A complex number z is said to be* algebraic *if there are integers $a_0, \ldots, a_n$, not all zero, such that*

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Hint 2: the above hint sucks.
This is equivalent to Problem 6.
Really, we did all the work in Problems 4, 5 and 6 above. Let P_n be the set of all polynomials of degree n and define $f_n : \mathbb{Z}^{n+1} \times \mathbb{N} \to P_n \times \{1, 2, 3, \ldots, n\}$ Since the Fundamental Theorem of Algebra still applies we can apply the same argument in 6 to show that $P_n \times \{1, 2, 3, \ldots, n\}$ is countable. The set of all algebraic numbers, which is the set $\bigcup_{n=1}^{\infty} P_n \times \{1, 2, 3, \ldots, n\}$ is then a countable union of countable sets, and so is itself countable.
3. *Prove that there exist real numbers which are not algebraic.*
Let $\mathbb{A}$ denote the set of algebraic numbers. By Theorem 2.43, $\mathbb{R}$ is uncountable. But $\mathbb{A}$ is countable. Therefore $\mathbb{A}$ cannot be
which means there must exist numbers in $\mathbb{R}$ that are not in $\mathbb{A}$; that is, numbers which are not algebraic.
4. *Is the set of all irrational real numbers countable?*
No they are not, by a similar argument to Problem 3. The real numbers is the union of $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ (that is, the rationals and irrationals). Since $\mathbb{Q}$ is countable, the irrationals being countable would make $\mathbb{R}$ the union of two countable sets, and thus itself countable. Since $\mathbb{R}$ is uncountable, at least one of the rationals and irrationals must be uncountable, and since $\mathbb{Q}$ is countable the irrationals must not be.