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Homework: Vol. 3

5. *Construct a bounded set of real numbers with exactly three limit points.*
Consider the set $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$; that is, $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. I claim that $p = 0$ is a limit point. Let $1 > \varepsilon > 0$. Then $\frac{1}{p+\varepsilon} = \frac{1}{\varepsilon} > 1$, and by the Archimedian Principle there exists $N \in \mathbb{N}$ such that $N > \frac{1}{p+\varepsilon}$. Then $\varepsilon > p + \frac{1}{N} > p$, and so $p + \frac{1}{N} \in N_\varepsilon(p)$.

That's one limit point. But there's nothing stopping us from joining three of these together; let's say $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} \cup \left\{1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, \ldots\right\} \cup \left\{2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots\right\}$. Using a similar argument to the above (this is why I was using p everywhere) this set has limit points at 0, 1 and 2. Note further that these sets are disjoint, so they cannot create new limit points. Finally, A is clearly bounded by $\frac{5}{2}$, so we have a set that fits the bill.

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6. *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?*

- Suppose $x \in E'^c$. Then x is not a limit point of E. This means that there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \cap (E \setminus \{x\}) = \varnothing$. Let y be a point in this neighborhood, and let $\alpha = d(x, y)$ (note that $\alpha < \varepsilon$). Since neighborhoods are open, there exists a neighborhood $N_{\varepsilon'}(y) \subset N_\varepsilon(x)$. In this neighborhood we've already established that there are no points of E. This means that y is not a limit point of E.

  We thus conclude that $N_\varepsilon(x)$ cannot have any limit points in it; this means that $N_\varepsilon(x) \subset E'^c$. Therefore E'^c is open, and E' consequently closed.

- We want to show that E and $\overline{E}$ have the same limit points. Suppose x is a limit point in E; that is $x \in E'$ so $x \in \overline{E}$. Then any $N_\varepsilon(x)$ has a point $y \in E$ such that $y \neq x$. But this point must also be in $\overline{E}$, so x is a limit point of $\overline{E}$.

  Conversely, suppose x is a limit point of $\overline{E}$. Then any $N_\varepsilon(x)$ has infinitely many points either in E or E'. Suppose every point in $N_\varepsilon(x)$ is in E'; that is $N_\varepsilon(x) \cap (E - \{x\}) = \varnothing$, and let y be one of those points (hence $y \in E'$). But $N_\varepsilon(x)$ is open, so y is in the interior of $N_\varepsilon(x)$; this means that there exists $\varepsilon' > 0$ such that $N_{\varepsilon'}(y) \subset N_\varepsilon(x)$. This means that $N_{\varepsilon'}(y) \cap E$ can contain *at most* x, a finite number. This means that y cannot be a limit point of E, directly contradicting our assumption that it was. Thus our supposition that $N_\varepsilon(x) \cap (E - \{x\}) = \varnothing$ is false; the must be a point in $N_\varepsilon(x)$ also in E. This means that x is a limit point of E.

- Do E and E' always have the same limit points? The way the question is worded suggests not, but surprisingly the discrete space won't be the one to throw a wrench in it. Consider the set $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. As shown in Problem 5, $E' = \{0\}$. But this is a finite point set, which therefore has no limit points.

7. *Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.*

(a) *If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$ for $n = 1,2,3,\ldots$.*

Since $\bigcup_{i=1}^n \overline{A_i}$ is closed and $B = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}$, we have $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

(b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A_i}$.
Let $x \in \bigcup_{i=1}^\infty \overline{A_i}$.

Then$x \in \overline{A_k}$ for some $k \in \mathbb{N}$, so $x \in A_k$ or $x \in A_k'$.

- $x \in A_k$ then$x \in B$, so $x \in \overline{B}$.
- If $x \in A_k'$ then any $N_\varepsilon(x)$ contains infinitely many points of A_k. Therefore it also contains infinitely many points of B. Thus we have $x \in B'$ and $x \in \overline{B}$.

Show, by an example, that this inclusion can be proper.
Consider $A_1, A_2, \ldots$ such that $A_k = \left\{\frac{1}{k}\right\}$. Then $\bigcup_{i=1}^n A_i$ devolves into the set given in problem 5. This has 0 as a limit point, but it's not the limit point of any A_k.

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8. *Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.*
This is true for all open sets in $\mathbb{R}^2$ (proper or not so I don't know why the distinction is being made). If A is an open set and $x \in A$, then any $B(x;r) \in A$ (and we know there is one because A is open) contains infinitely (in fact uncountably) many points. Thus everything in A is a limit point.

However, this isn't true for closed subsets of $\mathbb{R}^2$. If we look at the set $\{(0,0)\}$, it has no limit points and in particular the one point in the set isn't a limit point.

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9. *Let E^o denote the set of all interior points of a set E.*

(a) *Prove that E^o is always open.*
Let $p \in E^o$. Then $\exists B(p,r) \subseteq E$. Let $q \in B(p,r)$. Then there exists s such that $0 < s < r - d(p,q)$ and $B(q,s) \subseteq B(p,r) \subseteq E$. Therefore $q \in E^o$, and since q was arbitrary in $B(p,r)$ we have $B(p,r) \subseteq E^o$ Since we can do this for anything in E^o, it must be open.

(b) *Prove that E is open if and only if $E^o = E$.*
$\Rightarrow$ It is always true that $E^o \subseteq E$. If E is open, then by definition $E \subseteq E^o$, and therefore $E = E^o$.

$\Leftarrow$ If $E = E^o$, by (a) E^o is open and therefore E is open.

(c) *If $G \subset E$ and G is open, prove that $G \subset E^o$.*

Let $p \in G$. We want to show that $p \in E$. Since G is open and $G \subseteq E$, $\exists B(p,r) \subseteq G \subseteq E$. Therefore $p \in E^o$.

(d) *Prove that the complement of E^o is the closure of the complement of E.*
In other words, $(E^o)^c = \overline{E^c}$.

$\subseteq$: Let $x \in (E^o)^c$. If $x \in (E)^c$ then clearly $x \in \overline{(E)^c}$. If $x \in E$, then any neighborhood $N_\varepsilon(x)$ is not a subset of E. Thus there exists $y \in N_\varepsilon(x)$ such that $y \in (E)^c$. Since $x \neq y$, we have x is a limit point of E^c and $x \in \overline{E^c}$.

$\supseteq$: Let $x \in \overline{E^c}$. If $x \in E^c$, then again $x \in (E^o)^c$ is immediate. If $x \in E$, then for any $\varepsilon > 0$, $N_\varepsilon(r)$ contains a point $x \neq y$ such that $y \in E^c$. Thus x is not an interior point of E. This means that $\overline{E^c} \subseteq (E^o)^c$.

(e) *Do E and $\overline{E}$ always have the same interiors?*
No. Consider in $\mathbb{R}^2$ the set $\{(x,y)\,|\,0 < d(x,y) < 1\}$. $0 \notin E^o$, but $0 \in \overline{E}$ since it's the closed ball of radius 1 around $(0,0)$.

(f) *Do E and E^o always have the same closures?*
No. Consider $E = \mathbb{N}$ in $\mathbb{R}$. Since $\mathbb{N}$ is a bunch of isolated points in $\mathbb{R}$, the interior is nothing; that is, $\mathbb{N}^o = \varnothing$. But $\overline{\mathbb{N}} = \mathbb{N}$ and $\overline{\varnothing} = \varnothing$.

10. *Let X be an infinite set. For $p \in X$ and $q \in X$, define*

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

- First let's prove that it's a metric.

  (a) Clearly $d(p,q) > 0$ if $p \neq q$ and $d(p,q) = 0$ if $p = q$, because that's how we defined it.

  (b) $d(p,q) = d(q,p)$ is similarly clear.

  (c) If $p = q$ then

  $$\begin{aligned} d(p,q) &= 0 \\ &\leq 0+0 \\ &\leq d(p,r)+d(q,r). \end{aligned}$$

  If $p \neq q$, then either $p \neq r$ or $q \neq r$; assume without loss of generality the former. Then $d(p,q) = 1$ and

  $$d(p,r)+d(q,r) = 1+d(q,r) \leq 1 = d(p,q).$$

- Every set is open; if A is a set in X and $x \in A$, then the ball of radius $\frac{1}{2}$ around x is just x, which is in A. Thus everything in A has a neighborhood around it contained in A, so $x \in A^o$. Since $A^o = A$, A is open.

- Furthermore, every set is also closed. If $A \subseteq X$ then $A^c \subseteq X$ and A^c is open (because every set is open), so $(A^c)^c = A$ is closed.

- Finite subsets of X are clearly compact. If a set A is infinite, however, it is not compact. For every $x \in A$, let $G_x = B\left(r; \frac{1}{2}\right)$. Then

  $$\bigcup_{x \in A} G_x$$

  is a cover for A, but it cannot be reduced as removing and G_x uncovers x.

11. *For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define*

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= \left|x^2-y^2\right| \\ d_4(x,y) &= |x-2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

(a) Surprisingly (to me at least), d_1 is not a metric space, for it violates the triangle inequality. If $x=1$, $y=2$ and $z=3$ then $d(d,z)=4$, but $d(x,y)+d(y,z)=2$.

(b) d_2 is a metric. We need to show the three properties defining a metric space.

2 If $x \neq y$, then $|x-y|>0$, so $\sqrt{|x-y|}>0$; also $d(x,x)=\sqrt{|x-x|}=\sqrt{0}=0$.

2 $d(x,y)=\sqrt{|x-y|}=\sqrt{|y-x|}=d(y,x)$.

2 Let $x,y,z \in \mathbb{R}$. Then

$$\begin{aligned} |x-y| &\leq |x-z|+|y-z| \\ d(x,y)=\sqrt{|x-y|} &\leq \sqrt{|x-z|+|y-z|} \\ &\leq \sqrt{|x-z|}+\sqrt{|y-z|} \\ &= d(x,z)+d(y,z). \end{aligned}$$

(c) d_3 is not a metric space, for $d(1,-1)=0$ but $1 \neq -1$.

(d) d_4 is not a metric space because $d(1,3)=5$ but $d(3,1)=1$.

(e) d_5 is a metric space. What I'll do is show that if $d(x,y)$ is a metric space, then $d'(x,y)=\frac{1}{1+d(x,y)}$ is a metric space, and d_5 will be a corollary.

5 If $\frac{1}{1+d(x,y)}=0$, then $d(x,y)=0$, which implies that $x=y$.

5 $d'(x,y)=\frac{1}{1+d(x,y)}=\frac{1}{1+d(y,x)}=d'(y,x)$.

5 We need to show that $d'(x,z) \leq d'(x,y)+d'(y,z)$; that is,

$$\frac{1}{1+d(x,z)} \leq \frac{1}{1+d(x,y)}+\frac{1}{1+d(y,z)}$$

Let $\alpha = d(x,z)$, $\beta = d(x,y)$ and $\gamma = d(y,z)$. Then with some manipulation we have

$$\begin{aligned}
\alpha &\leq \beta+\gamma \\
\alpha &\leq \beta+\gamma+2\beta\gamma+\alpha\beta\gamma \text{ (because each term is positive)} \\
\alpha+\alpha\beta+\alpha\gamma+\alpha\beta\gamma &\leq (\beta+\alpha\beta+\beta\gamma+\alpha\beta\gamma)+(\gamma+\alpha\gamma+\beta\gamma+\alpha\beta\gamma) \\
\alpha(\beta+1)(\gamma+1) &\leq \beta(\gamma+1)(\alpha+1)+\gamma(\alpha+1)(\beta+1) \\
\frac{\alpha}{1+\alpha} &\leq \frac{\beta}{1+\beta}+\frac{\gamma}{1+\gamma} \\
d'(x,z) &\leq d'(x,y)+d'(y,z).
\end{aligned}$$

That's a lot of work, but now we have our claim. Now, since we know that $\mathbb{R}$ under $d(x,y) = |x-y|$ is a metric, we have

$$d'(x,y) = \frac{|x-y|}{1+|x-y|}$$

is also a metric.

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12. *Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$ for $n = 1,2,3,\ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).*

Let $C = \bigcup_{\alpha\in A} G_\alpha$ be an open cover of K. At least one G_0 covers 0 (or else C cannot be a cover); this means there is $G_0 \in C$ such that $0 \in G_0$. Since G_0 is open, there exists $r > 0$ such that $(-r,r) \subseteq G_0$. Since by the Archimedian Principle there are only a finite number of n such that $r < \frac{1}{n}$, there are only a finite number of points in K that are uncovered. Select one cover for each remaining point, and we have a finite subcover.

Since we can do this for any subcover C, K is compact.

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13. *Construct a compact set of real numbers whose limit points form a countable set.*

For $k \in \mathbb{N}$, Let

$$D_k = \left\{\frac{1}{10}\right\}^k \cup \left\{\left(\frac{1}{10}\right)^k + \left(\frac{1}{10}\right)^m\right\}_{m=k+1}^{\infty}$$

and let $D = \{0\} \cup \bigcup_{k\in\mathbb{N}} D_k$. Thus $D_1 = \{.1, .11, .101, \ldots\}$ while $D_2 = \{.01, .011, .0101, \ldots\}$. and so on. Using a similar argument to problem 5, The limit points are $0, \frac{1}{10}, \frac{1}{10^2}, \ldots = \bigcup_{i=0}^{\infty}\left(\frac{1}{10}\right)^n$ which is countable, and nothing else (because each D_k is disjoint from the other sets). Thus the set of limit points is countable.

We're not hog-tied by having to use the definition this time, so all we have to do to show that D is compact is show that it's closed and bounded. Clearly A is bounded above by 2 and below by 0 so we're in the clear there. Furthermore, D contains all its limit points because we expressly included them, so D is closed. Since D is closed and bounded, it is compact.

14. *Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.*

Consider $A = \left(\frac{1}{2},1\right) \cup \left(\frac{1}{3},1\right) \cup \left(\frac{1}{4},1\right) \cup \cdots$; that is, $A_n = \bigcup_{n=1}^{\infty} \left(\frac{1}{n},1\right)$. This is a cover for $(0,1)$ because for any $\alpha \in (0,1)$ we have $\alpha > \frac{1}{n}$ for some n by Archimedes, so $\alpha \in \left(\frac{1}{n},1\right) \in A_n$.

Let $G = \bigcup_{i=1}^{n} A_{k_i}$ such that $k_1 < k_2 < \ldots < k_n$. Since $A_1 \supset A_2 \supset A_3 \supset \cdots$, there is a "maximum" cover $\left(\frac{1}{k_n},1\right)$ such that $0 < \frac{1}{k_n}$ and so, for example, $\frac{1}{2M} \notin \bigcup_{i=1}^{n} A_k$.

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15. *Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$ for example) if the word "compact" is replaced by "closed" or by "bounded."*

- If $\{K_\alpha\}$ is a collection of **closed** subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then is $\bigcap K_\alpha$ nonempty?

  Consider $K_x = [x,\infty)$ for $x \in \mathbb{R}^+$. Clearly any finite subcollection $A = \bigcap_{i=1}^{n} K_{k_i}$ is nonempty, because if α is the maximum of the k_i's then $\alpha+1$ is in every set. But for any $\alpha \in \mathbb{R}^+$, $\alpha \notin A_{\alpha+1}$ and so $x \notin \bigcap K_\alpha = \varnothing$.

- If $\{K_\alpha\}$ is a collection of **bounded** subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then is $\bigcap K_\alpha$ nonempty?

  Let $K_x = (x,1)$ for $x \in (0,1)$. Then $A \neq \bigcap_{i=1}^{n} K_{k_i}$ is nonempty, because if α is the minimum of the k_i's then $\frac{\alpha}{2}$ is in every set of the collection. However, $\bigcap K_\alpha$ is empty, because clearly anything not in $(0,1)$ is not in $\bigcap K_\alpha$, and for any $\alpha \in (0,1)$ we know $\alpha \notin K_{a/2}$.

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22. *A metric space is called* separable *if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. Hint: Consider the set of points which have only rational coordinates.*

Consider $\mathbb{Q}^k \subset \mathbb{R}^k$. This is countable (as we showed last homework). We need to show that $\overline{\mathbb{Q}^k} = \mathbb{R}^k$.

Let $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ and suppose it's not in $\mathbb{Q}^k$ (otherwise $x \in \overline{\mathbb{Q}^k}$ follows immediately). Let $\varepsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that for each x_i, there exists $q_i \in \mathbb{Q}$ such that $|x_i - q_i| < \frac{\varepsilon^2}{k}$. Let $q = (q_1, q_2, \ldots, q_k)$. Then

$$\begin{aligned} d(x,q) &= \sqrt{\sum_{i=1}^{k} |x_i - q_i|} \\ &\leq \sqrt{k \cdot \frac{\varepsilon^2}{k}} \\ &\leq \varepsilon. \end{aligned}$$

Since q cannot equal x (because q is in $\mathbb{Q}^k$ and x is not), x is a limit point of $\mathbb{Q}^k$. Thus $x \in \mathbb{R}^k$ is also in $\mathbb{Q}^k \cup \left(\mathbb{Q}^k\right)'$; that is, $\mathbb{R}^k = \overline{\mathbb{Q}^k}$. Thus $\mathbb{Q}^k$, a countable subset of $\mathbb{R}^k$, is dense in $\mathbb{R}^k$, and so $\mathbb{R}^k$ is separable.

23. *A collection* $\{V_\alpha\}$ *of open subsets of* X *is said to be a* base *for* X *if the following is true: For every* $x \in X$ *and every open set* $g \subset X$ *such that* $x \in G$*, we have* $x \in V_\alpha \subset G$ *for some* α*. In other words, every open set in* X *is the union of a subcollection of* $\{V_\alpha\}$.

Prove that every separable metric space has a countable *base. Hint: Take all neighborhoods with rational radius and center in some countable dense subset of* X.

X is separable so it has a countable dense subset. Let's call it Q. Let $\mathcal{F}$ be the set of open balls with centers in Q and radius in $\mathbb{Q}$; that is, let $\mathcal{F} = \left\{B_q(x) : q \in \mathbb{Q} \text{ and } x \in Q\right\}$. We want to show that $\mathcal{F}$ is a base for X.

Let $x \in X$ and G be an open subset of X such that $x \in G$. Then there exists $r > 0$ such that $N_r(x) \subseteq G$. Since Q is dense in X, there exists a sequence $\{q_n\}$ in Q such that $\{q_n\} \to x$. Let r_n be such that $r_n < \frac{r}{3}$, and $G_\alpha \in \mathcal{F}$ such that $N_{r_n}(q_n) \subseteq N_r(x)$ for large enough n. Thus we have $x \in G_\alpha \in \mathcal{F}$ for some α, which is what we need to show that $\mathcal{F}$ is a base for X.

24. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.*

Let $\delta > 0$ and $x_1 \in X$. For any $x_i \in X$, choose x_{i+1} such that $d(x_i, x_j) \geq \delta$ for all $j = 1, 2, \ldots, i$.

Suppose this process never stops. Then we have an infinite sequence $\{x_i\}_{i=1}^{\infty}$ such that every point is at least δ away from every other point. This means that the sequence can never converge to anything. But X is limit-compact; which means the sequence is supposed to have a limit point. This is a contradiction, and we must conclude that the process must indeed stop; that is, the process yields a *finite* set of points $A_\delta = \{x_1, x_2, \ldots, x_{i_\delta}\}$. Note that the specific value of i_δ (that is, the number of elements in A_δ) doesn't depend strictly on δ; it can vary depending on our specific choices of each x_i. Note further that $\bigcup_{i=1}^{n} N_\delta(x_i)$ is a cover (indeed a finite cover) for X, because if there exists $x \notin A_\delta$, then $N_\delta(x)$ has no points of A_δ in it, and we can assign x as a new element of A_δ in the above process. Finally, note that since our only restriction on δ is that it's positive and fixed, we can create a new A_δ for any (fixed) positive number we choose.

For each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ and let $A_n = \{x_1, x_2, \ldots, x_{\delta_i}\}$ be a set determined as above, and let G_n be the union of the set of δ-neighborhoods around the x_i's. Let $A = \bigcup_{i=1}^{n} A_i$; that is, A is the set of centers of δ-balls for all $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. I claim that A is dense in X.

Let $z \in X$. If $z \in A$ then $z \in \overline{A}$. Suppose $z \notin A$ (that is, z is not the center of any δ-neighborhood), let $r > 0$, and consider $N_r(z)$. Note that $N_{r/2}(z)$ is covered by a finite number of neighborhoods $N_\alpha(x_1), \ldots, \ldots N_\alpha(x_k)$ of radius $\frac{1}{2/r+1}$ and each $x_i \in A$ for $i = 1, \ldots k$. Let $y \in N_{r/2}(z) \cap N_\alpha(x_1)$. Then

$$\begin{aligned} d(x_1, z) &\leq d(x_1, y) + d(y, z) \\ &\leq \frac{r}{2} + \alpha \\ &\leq r. \end{aligned}$$

Therefore, $x_1 \in N_r(z)$ but $z \neq x_1$ (because $z \notin A$ but $x_1 \in A$). This means that z is a limit point of A.

Since z was arbitrary in X, we have that *every* point in X is either in A or a limit point of A. That is, $X = A \cup A' = \overline{A}$, so A, a countable set (it's the countable union of finite sets) is dense in X, and so finally X is separable. Whew!

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25. *Prove that every compact metric space K has a countable base, and that K is therefore separable. Hint: For every positive integer n, there are finitely many neighborhood of radius $1/n$ whose union covers K.*

Suppose K is compact. Then by what we proved in class it's limit point compact, so by Problem 24 it's separable and by Problem 23 it has a countable base.

26. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. Hint: By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a* countable *subcover* $\{G_n\}$, $n = 1, 2, 3, \ldots$. *If no finite subcollection of* $\{G_n\}$ *covers X, then the complement* F_n *of* $G_1 \cup \cdots \cup G_n$ *is nonempty for each n, but* $\bigcap F_n$ *is empty. If E is a set which contains a point from each* F_n*, consider a limit point of E, and obtain a contradiction.*

Let X be a limit compact metric space. By the two exercises above, X has a countable base. This means that it has a countable subcover, $\{G_n\}$. Suppose X is not compact. Then there is no *finite* subcollection of $\{G_n\}$ that covers X. For any $n \in \mathbb{N}$, let $F_n = (\bigcup_{i=1}^{n} G_n)^c$; for example, $F_1 = G_1^c$, $F_2 = (G_1 \cup G_2)^c = G_1^c \cap G_2^c$ and so on. Then F_n is always nonempty (otherwise the G_n's would form a cover), but $\bigcap_{i=1}^{\infty} F_n$ is empty. Let $E = \{e_1, e_2, e_3 \ldots\}$ be a set that contains a point from each F_n (that is, $e_1 \in F_1$, $e_2 \in F_2$ and so on), and let p be a limit point of E.

Since p is a limit point of E, there is a sequence $\{e_\alpha\} = \{e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}, \ldots\}$ in E (which remember is a subset of F_1) that converges to p. Note that for any $k \geq 2$, $F_{k-1} \subseteq F_k$. Therefore every F_k contains a subsequence of $\{e_\alpha\}$, which means that p is also a limit point of every F_k. Furthermore, each F_n is closed (because it's the complement of a finite union of open sets). This means that $p \in F_k$ for any F_k; that is, $p \in \bigcap F_k$.. But $\bigcap F_n$ was supposed to be empty. This is a contradiction.

Therefore, we must reject the assumption that X lacks a finite subcover of $\{G_n\}$ that covers X, which means X must be compact.