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Math <Course>

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Homework: Vol. 4

Spivak

4.

(a) *Prove that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence.*

(b) *Prove that any subsequence of a convergent sequence converges.*

5.

(a) *Prove that if* $0 < a < 2$*, then* $a < \sqrt{2a} < 2$.

(b) *Prove that the sequence*

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converges.

(c) *Find the limit. Hint: Notice that if* $\lim_{n\to\infty} a_n = l$*, then* $\lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2l}$*, by Theorem 1.*

6. *Let* $0 < a_1 < b_1$ *and define*

$$a_{n+1} = \sqrt{a_n b_n}, \, b_{n+1} = \frac{a_n + b_n}{2}.$$

(a) *Prove that the sequences* $\{a_n\}$ *and* $\{b_n\}$ *each converge.*

(b) *Prove that they have the same limit.*

8. *Idenitfy the function* $f(x) = \lim_{n\to\infty} \left(\lim_{k\to\infty} (\cos n!\pi x)^{2k} \right)$. *(It has been mentioned many times in this book.)*

9. *Many impressive looking limits can be evaluated easily, because they are really lower or upper sums in disguise. With this remark as hit, evaluate each of the following.*

- **(iv).** $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2} \right)$.
- **(v).** $\lim_{n\to\infty} \left(\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(n+n)^2} \right)$.
- **(vi).** $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} \right)$.

10. *Although limits like* $\lim_{n\to\infty}\sqrt[n]{n}$ *and* $\lim_{n\to\infty}a^n$ *can be evaluated using facts about the behavior of the logarithm and exponential functions, this approach is vaguely dissatisfying, because integral roots and powers can be defined without using the exponential function. Some of the standard "elementary" arguments for such limits are outlined here; the basic tools are inequalities derived from the binomial theorem, notably*

$$(1+h)^n \geq 1+nh, for\, h>0;$$

and, for part (e),

$$(1+h)^n \geq 1+nh+\frac{n(n-1)}{2}h^2 \geq \frac{n(n-1)}{2}h^2, for\ h>0.$$

(a) *Prove that* $\lim_{n\to\infty}a^n=\infty$ *if* $a>1$*, by setting* $a=1+h$*, where* $h>0$*.*

(b) *Prove that* $\lim_{n\to\infty}a_n=0$ *if* $0<a<1$*.*

(c) *Prove that* $\lim_{n\to\infty}\sqrt[n]{a}=1$ *if* $a>1$*, by setting* $\sqrt[n]{a}=1+h$ *and estimating* h*.*

(d) *Prove that* $\lim_{n\to\infty}\sqrt[n]{a}=1$ *if* $0<a<1$*.*

(e) *Prove that* $\lim_{n\to\infty}\sqrt[n]{n}=1$*.*

11.

(a) *Prove that a convergent sequence is always bounded.*

(b) *Suppose that* $\lim_{n\to\infty}a_n=0$*, and that each* $a_n>0$*. Prove that the set of all numbers* a_n *actually has a maximum member.*

12.

(a) *Prove that*

$$\frac{1}{n+1}<\log(n+1)-\log n<\frac{1}{n}.$$

(b) *If*

$$a_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\log n,$$

show that the sequence $\{a_n\}$ *is decreasing, and that each* $a_n\geq 0$*.*

13.

(a) *Suppose that* f *is increasing on* $[1,\infty)$*. Show that*

$$f(1)+\cdots+f(n-1)<\int_1^n f(x)\,dx<f(2)+\cdots+f(n).$$

(b) *Now choose* $f=\log$ *and show that*

$$\frac{n^n}{e^{n-1}}<n!<\frac{(n+1)^{n+1}}{e^n}.$$

16. *Prove that if* $\lim_{n\to\infty}a_n=l$*, then*

$$\lim_{n\to\infty}\frac{(a_1+\cdots a_n)}{n}=l.$$

Hint: This problem is very similar to (in fact is its a special case of) Problem 13-40.

17. *Suppose that* f *is continuous and* $\lim_{x\to\infty}f(x+1)-f(x)=0$*. Prove that* $\lim_{x\to\infty}f(x)/x=0$*. Hint: See the previous problem.*

18. *Suppose that* $a_n>0$ *for each* n *and that* $\lim_{n\to\infty}a_{n+1}/a_n=l$*. Prove that* $\lim_{n\to\infty}\sqrt[n]{a_n}=l$*. Hint: This requires the same sort of argument that works in Problem 16, together with the fact that* $\lim_{n\to\infty}\sqrt[n]{a}=1$*, for* $a>0$*.*

22.

(a) *Use Problem 2-5 to show that if $c \neq 1$, then*

$$c^m + c^{m+1} + \cdots + c^n = \frac{c^m - c^{n+1}}{1-c}.$$

(b) *Suppose that $|c| < 1$. Prove that*

$$\lim_{m,n\to\infty} c^m + \cdots + c^n = 0.$$

(c) *Suppose that $\{x_n\}$ is a sequence with $|x_n - x_{n+1}| \leq c^n$, where $c < 1$. Prove that $\{x_n\}$ is a Cauchy sequence.*

23. *Suppose that f is a function on $\mathbb{R}$ such that*

$$|f(x) - f(y)| \leq c\,|x-y|$$

for all x and y, where $c < 1$. (Such a function is called a contraction.*)*

(a) *Prove that f is continuous.*

(b) *Prove that f has at most one fixed point.*

(c) *By considering the sequence*

$$x, f(x), f(f(x)), \ldots,$$

for any x, prove that f does have a fixed point.

24.

(a) *Prove that if f is differentiable and $|f'| < 1$, then f has at most one fixed point.*

(b) *Prove that if $|f'(x)| \leq c < 1$ for all x, then f has a fixed point.*

(c) *Give an example to show that the hypothesis $|f'(x)| \leq 1$ is not sufficient to insure that f has a fixed point.*

25. *This problem is a sort of converse to the previous problem. Let b_n be a sequence defined by $b_1 = a$, $b_{n+1} = f(b_n)$. Prove that if $b = \lim_{n\to\infty} b_n$ exists and f' is continuous at b, then $|f'(b)| \leq 1$. Hint: If $|f'(b)| > 1$, then $|f'(x)| > 1$ for all x in an interval around b, and b_n will be in this interval for large enough n. Now consider f on the interval $[b, b_n]$.*

27. *Let $\{x_n\}$ be a sequence which is bounded, and let*

$$y_n = \sup\{x_n, x_{n+1}, x_{n+1}, \ldots\}.$$

(a) *Prove that the sequence $\{y_n\}$ converges. The limit $\lim_{n\to\infty} y_n$ is denoted by $\overline{\lim_{n\to\infty}} x_n$ or $\lim_{n\to\infty} \sup x_n$, and is called the* ***limit superior****, or* ***upper limit****, of the sequence $\{x_n\}$.*

(b) *Find $\overline{\lim_{n\to\infty}} x_n$ for each of the following:*

2 $x_n = \frac{1}{n}$.
2 $x_n = (-1)^n \frac{1}{n}$.
2 $x_n = (-1)^n \left[1 + \frac{1}{n}\right]$.
2 $x_n = \sqrt[n]{n}$.

(c) *Define $\lim_{n\to\infty} \inf x_n$ and prove that*

$$\lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n.$$

(d) *Prove that $\lim_{n\to\infty} x_n$ exists if and only if $\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \inf x_n$ and that in this case $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \inf x_n$.*

(e) *Recall the definition, in problem 8-18, of $\overline{\lim} A$ for a bounded set A. Prove that if the numbers x_n are distinct, then $\lim_{n\to\infty} \sup x_n = \overline{\lim} A$, where $A = \{x_n : n \in \mathbb{N}\}$.*

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1. *Prove that convergence of* $\{s_n\}$ *implies convergence of* $\{|s_n|\}$. *Is the converse true?*

2. *Calculate* $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

3. *If* $s_1=\sqrt{2}$, *and*

$$s_{n+1}=\sqrt{2+\sqrt{s_n}}\ (n=1,2,3,\ldots),$$

prove that $\{s_n\}$ *converges, and that* $s_n<2$ *for* $n=1,2,3,\ldots$.

4. *Find the upper and lower limits of the sequence* $\{s_n\}$ *defined by*

$$\begin{aligned} s_1 &= 0 \\ s_{2m} &= \frac{s_{2m-1}}{2} \\ s_{2m+1} &= \frac{1}{2}+s_{2m}. \end{aligned}$$

5. *For any two real sequences* $\{a_n\}$, $\{b_n\}$, *prove that*

$$\lim_{n\to\infty}\sup(a_n+b_n)\le\lim_{n\to\infty}\sup a_n+\lim_{n\to\infty}\sup b_n,$$

provided that sum on the right is not of the form $\infty-\infty$.

6. *Investigate the behavior (convergence or divergence) of* $\sum a_n$ *if*

(a) $a_n=\sqrt{n+1}-\sqrt{n}$

(b) $a_n=\frac{\sqrt{n+1}-\sqrt{n}}{n}$

(c) $a_n=(\sqrt[n]{n}-1)^n$

7. *Prove that the convergence of* $\sum a_n$ *implies the convergence of*

$$\sum\frac{\sqrt{a_n}}{n},$$

if $a_n\ge 0$.

8. *If* $\sum a_n$ *converges, and if* $\{b_n\}$ *is monotonic and bounded, prove that* $\sum a_nb_n$ *converges.*

9. *Find the radius of convergence of each of the following power series:*

(a) $\sum n^3z^n$

(b) $\sum\frac{2^n}{n!}z^n$

10. *Suppose that the coefficients of the power series* $\sum a_nz^n$ *are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.*

11. *Suppose* $a_n>0$, $s_n=a_1+\cdots+a_n$, *and* $\sum a_n$ *diverges.*

(a) *Prove that* $\sum\frac{a_n}{1+a_n}$ *diverges.*

(b) *Prove that*

$$\frac{a_{N+1}}{s_{N+1}}+\cdots+\frac{a_{N+k}}{s_{N+k}}\ge 1-\frac{s_N}{s_{N+k}}$$

and deduce that $\sum\frac{a_n}{s_n}$ *diverges.*

(c) *Prove that*

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ *converges.*

(d) *What can be said about*

$$\sum \frac{a_n}{1+na_n} \quad \text{and} \quad \sum \frac{a_n}{1+n^2 a_n}?$$

12. *Suppose* $a_n > 0$ *and* $\sum a_n$ *converges. Put*

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) *Prove that*

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, *and deduce that* $\sum \frac{a_n}{r_n}$ *diverges.*

(b) *Prove that*

$$\frac{a_n}{\sqrt{r^n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ *converges.*

13. *Prove that the Cauchy product of two absolutely convergent series converges absolutely.*

14. *If* $\{s_n\}$ *is a complex sequence, define its arithmetic means* σ_n *by*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \ (n = 0, 1, 2, \ldots).$$

(a) *If* $\lim s_n = s$, *prove that* $\lim \sigma_n = s$.

(b) *Construct a sequence* $\{s_n\}$ *which does not converge, although* $\sigma_n = 0$.

(c) *Can it happen that* $s_n > 0$ *for all n and that* $\limsup s_n = \infty$, *although* $\lim \sigma_n = 0$?