

October 2, 2006

**Theorem** Let  $A$  be the set of all 0-1 sequences. Then  $A$  is uncountable (Cantor's diagonal process)

**Proof.** We first show that every countable subset of  $A$  must be proper. Given a countable subset  $E$  of  $A$ . Then  $E$  can be enumerated as  $a_1 = (a_{11}, a_{12}, a_{13}, \dots)$ ,  $a_2 = (a_{21}, a_{22}, a_{23}, \dots)$  and so on. We construct a 0-1 sequence  $b = \{b_1, b_2, \dots\}$  as follows:

For each  $n = 1, 2, 3, \dots$ ,

$$b_n = \begin{cases} 1 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} = 1. \end{cases}$$

Then  $b$  is a 0-1 sequence and  $b \notin E$ . Why? Because  $b$  is different from  $a_n$  in at least the  $n$ th position. Therefore  $E$  is a proper subset of  $A$ , and so  $A$  is uncountable (because if  $A$  were countable then  $A$  would be a countable subset of  $A$ , and by the first part of the proof, we know  $A$  must be a proper subset of  $A$ .)

**Theorem** Prove that  $[0, 1)$  is uncountable.

**Proof.** Every  $x \in [0, 1)$  has a binary expansion

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n},$$

where  $x_i \in \{0, 1\}$ . However some  $x \in [0, 1)$  may have two binary expansions. For example,  $\frac{1}{2} = 0.1\bar{0} \dots = 0.0\bar{1}$ . To avoid the one-to-two, we consider  $E = A \setminus B$ , where  $A$  is the set of all binary expansions and  $B$  is the set of all binary expansions that terminate in ones. Note that  $A$  is uncountable and  $B$  is countable (practice exercise). Therefore  $E = A \setminus B$  is uncountable. Since every  $x \in [0, 1)$  has a unique binary expansion,

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n},$$

where  $\{x_1, x_2, \dots, x_n\}$  does not terminate in ones. Therefore there is a 1-1 function  $f$  from  $[0, 1)$  onto  $E$ ,  $f(x) = (x_1, \dots, x_n, \dots)$  where  $x \in [0, 1)$ . Since  $E$  is uncountable, so is  $[0, 1)$ .

For decimal expansions:

Let  $A$  be the set of all expansions and let  $B$  be the set of decimal expansions that terminate in nines. Then  $A$  is uncountable (we know how) and  $B$  is countable (exercise) and  $E = A \setminus B$  is uncountable; in fact,  $E \sim A$ . For every  $x \in [0, 1)$ ,  $x$  has a *unique* decimal expansion

$$x = \sum_{n=1}^{\infty} \frac{x_n}{10^n},$$

where  $(x_1, \dots, x_n)$  does not terminate in nines. Therefore  $[0, 1) \sim E$  and  $[0, 1)$  is uncountable.

**Theorem**  $(0, 1) \sim \mathbb{R}$ .

**Proof.**  $f(x) = \frac{2x-1}{x(x-1)}$ . Check that  $f' < 0$  for all  $x$  so it's 1-1 and we're done. An easier one is the tangent function:

$$f(x) = \tan \left[ \pi \left( x - \frac{1}{2} \right) \right]$$

**Theorem**  $(0, 1) \sim (0, 1]$ .

**Proof.** We shift each  $\frac{1}{n}$  to  $\frac{1}{n-1}$ :  $\frac{1}{2} \rightarrow 1$ ,  $\frac{1}{3} \rightarrow \frac{1}{2}$  and so on. Then you can use this and adapt it to show that  $[0, 1) \sim [0, 1]$ .

## Midterm

5 questions, some with multiple parts, he doesn't know if it'll be enough or too much. Read lecture notes from first chapter and section 2.1. It's not a memory test, but a test of understanding. The one we just did is a good problem.

But what about  $(0, 1)$  to  $(0, 1) \cup \{a_1, a_2, \dots\}$ ?

We'll do something similar, but merge them. Instead of just sending each  $\frac{1}{n}$  somewhere else, we'll do like we did when mapping  $\mathbb{N} \rightarrow \mathbb{Z}$ :

$$\left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \rightarrow \left\{ \frac{1}{2}, a_1, \frac{1}{3}, \dots \right\}$$

## Metric Spaces

**Definition.** Let  $X$  be a set. A **metric** or a **distance** is a function from  $X \times X$  to  $\mathbb{R}$  such that

- (a)  $d(x, y) \geq 0$  and  $d(x, y) = 0$  iff  $x = y$
- (b)  $d(x, y) = d(y, x)$
- (c)  $d(x, y) \leq d(x, z) + d(z, y)$

Note that we'll need the triangle inequality in almost every proof.  $(X, d)$  is called a **metric space**.

Examples:

- (a)  $(\mathbb{R}^1, d_1)$  is a metric space, where  $d_1(x, y) = |x - y|$ .
- (b)  $(\mathbb{R}^n, d)$  is a metric space where

$$\begin{aligned}d &= |\mathbf{x} - \mathbf{y}|, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \\ &= \sqrt{\sum_{i=1}^n (x_i - y_i)^2}\end{aligned}$$

**Proof of 2 $\Delta$ .** We proved earlier that  $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$

$$\begin{aligned}|\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}| |\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2\end{aligned}$$

so  $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ . Now we can say

$$\begin{aligned}|\mathbf{x} + \mathbf{y}| &= |\mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y}| \\ &\leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|\end{aligned}$$

**Example.** Let  $X$  be the set of all bounded real valued functions defined on  $[a, b]$ . Define  $d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$ . Now if  $f$  is bounded then  $|f(x)| \leq M \forall x \in [a, b]$ .  $E = \{f(x) : x \in [a, b]\}$  has a supremum and infimum.  $|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq 2M$  so the supremum exist. Suppose  $f_n(x) \rightarrow g(x)$ . Then we're getting uniform convergence.

Anyway,  $d$  is nonnegative since we're taking absolute values. Also  $d(f, g) = 0$  iff  $f \equiv g$ : let  $x \in [a, b]$ , then

$$|f(x) - g(x)| \leq \sup\{|f(x) - g(x)| : x \in [a, b]\} = d(f, g) = 0$$

Symmetric isn't a problem, but  $\Delta$  sucks, do it after midterm.