

7. Prove: if f is defined on \mathbb{R} and continuous at $x = 0$, and if $f(x_1 + x_2) = f(x_1) + f(x_2) \forall x_1, x_2 \in \mathbb{R}$, then f is continuous at all $x \in \mathbb{R}$.

Let's first investigate what $f(0)$ is. We know that $f(1)$ is defined, so

$$\begin{aligned}f(1+0) &= f(1) + f(0) \\f(1) &= f(1) + f(0) \\f(0) &= 0.\end{aligned}$$

f is also continuous at zero, so $\lim_{x \rightarrow 0} f(x) = 0$. That is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|y| < \delta$ implies $|f(y)| < \varepsilon$.

With this, we can turn to the task at hand. Let $a \in \mathbb{R}$. We need to show that $\lim_{x \rightarrow a} f(x) = f(a)$. Let's now look into manipulating $|f(x) - f(a)|$:

$$\begin{aligned}|f(x) - f(a)| &= |f(x) + f(-a) - f(-a) - f(a)| \\&= |f(x-a) - [f(a-a)]| \\&= |f(x-a) - 0| \\&= |f(x-a)|.\end{aligned}$$

But from what we showed above, if we let $y = x - a$ then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x-a)| = |f(x) - f(a)| < \varepsilon$. This means that $\lim_{x \rightarrow a} f(x) = f(a)$, and so f is continuous at a .