33. *Suppose that* $|f(x) - f(y)| \leq |x-y|^n$ *for* $n \geq 1$. *Prove that* f *is constant by considering* f'. *Compare with Problem 3-20.*

Are they missing an assumption that f is continuous and differentiable on an interval? If it's not continuous then I don't see how we can necessarily consider an f' that may not exist.

Anyway, I'll start off with a Lemma: If $|x-y| < 1$, then $\lim_{n\to\infty} |x-y|^n = 0$.

Proof: Let M be arbitrary. We want to find a value for n such that $|x-y|^n < M$. With some manipulation we have

$$\begin{aligned} \ln|x-y|^n &< \ln M \\ n\ln|x-y| &< \ln M \\ n &> \frac{\ln M}{\ln|x-y|} \end{aligned}$$

(Keep in mind that $|\alpha| < 1 \Rightarrow \ln|\alpha| < 0$.)

Without loss of generality, let $y > x$, and suppose $y \in (x, x+1)$. By the Mean Value Theorem there exists $c \in (x,y)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. But $\left|\frac{f(x)-f(y)}{x-y}\right| < |x-y|^{n-1}$. Since we can make $|x-y|^{n-1}$ arbitrarily small (remember that we restricted $|x-y|$ to be less than 1), we have

$$\begin{aligned} \left|\frac{f(x)-f(y)}{x-y}\right| &< 0 \\ \frac{f(x)-f(y)}{x-y} &= 0 \\ f(x) &= f(y). \end{aligned}$$

This can easily be extended to show that $f(x) = f(y)$ for any x and y.