Math 230A

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Homework, Vol. 1

Rudin

1. *If r is rational $(r \neq 0)$ and x is irrational, prove that $r+x$ and rx are irrational.*

Suppose $r+x$ is rational. Then $r+x=\frac{p}{q}$ for some $p,q\in\mathbb{Z}$. But then $x=\frac{p}{q}-r=\frac{p-rq}{q}$, implying that x is rational which is a contradiction. Similarly, assuming rx is rational leads to $x=\frac{p}{qr}$ for some $p,q\in\mathbb{Z}$ and thus rational.

2. *Prove that there is no rational number whose square is 12.*

Suppose there exist $p,q\in\mathbb{Z}$, $q\neq 0$ such that they are relatively prime and $\left(\frac{p}{q}\right)^2=12$. Then $p^2=12q^2=3(2q)^2$. This means that $3\mid p^2$ and $3\mid p$. Thus $9\mid p^2$ and $p^2=9k$ for some $k\in\mathbb{Z}$. $9k=3(2q)^2$ implies that $9\mid 3(2q)^2$, or $3\mid(2q)^2$. This means that 3 divides either 2^2 or q^2. We know $3\nmid 4$, so it must divide q^2. But this means that 3 divides both q and p, contradicting the assumption that they are relatively prime. Thus such p and q cannot exist, and so 12 cannot be the square of a rational number.

4. *Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha\leq\beta$.*

Since E is nonempty there's something in it; call it x. Since α is a lower bound of E it's at least as low as every thing in it; in particular $\alpha\leq x$. Similarly, β being an upper bound of E implies $x\leq\beta$, and a quick application of the transitive property yields $\alpha\leq\beta$.

5. *Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x\in A$. Prove that*

$$\inf A=-\sup(-A).$$

Since A is bounded below there exists by Theorem 1.11 $\inf A$, call it α. Then for all $x\in A$ we have $\alpha<x$ and consequently $-\alpha>-x$. But this means that $-\alpha$ is an upper bound of $-A$.

Now, suppose there is another upper bound of $-A$, call it $-\alpha'$, such that $-\alpha'\leq-\alpha$. Then $-x\leq-\alpha'$ for all $-x\in-A$ and consequently $x\geq\alpha'$ for all $x\in A$. Furthermore, $-\alpha'\leq-\alpha$ implies $\alpha'\geq\alpha$. But $\alpha=\inf A$, so $\alpha'=\alpha$ and $-\alpha'=-\alpha$. Thus $-\alpha=\sup(-A)$, or $\alpha=-\sup(-A)$.

8. *Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.*

Let $\mathbb{C}'$ be the set of complex numbers with some ordering applied, and suppose it is an ordered field. By 1.18d, $i^2=-1>0$ and by 1.18a, $-(-1)=1<0$. But this contradicts the second part of 1.18d (specifically, that the multiplicative identity is greater than the additive identity). Thus $\mathbb{C}'$ cannot be an ordered field.

9. *Suppose $z=a+bi$, $w=c+di$. Define $z<w$ if $a<c$, and also if $a=c$ but $b<d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of relation is called a* dictionary order*, or* lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?*

Let $S=\{a+bi\mid a<0\}$. This is bounded by, say, $1+0i$. Let $\alpha+\beta i$ be another upper bound of S. If $\alpha>0$, then $\alpha>\frac{\alpha}{2}>0$ and $\frac{\alpha}{2}$ is another, smaller upper bound of S. If $\alpha=0$, then $\alpha+(\beta-1)i$ is also another, smaller boundary of S. In either case, we can always find a smaller upper bound of S, so S has no least upper bound. Thus the Least Upper Bound Property does not hold.

13. *If x,y are real, prove that*

$$||x|-|y||\leq|x-y|.$$

We'll do two cases.

(a) If $|x| \geq |y|$ then $0 \leq |x|-|y|$, so $|x|-|y| \leq |x-y|$ implies that $||x|-|y|| \leq ||x-y|| = |x-y|$.

(b) If $|y| > x$ then $0 \leq |y|-|x|$, so $|x-y| = |y-x| \geq |y|-|x| \geq 0$, so we have $||y|-|x|| \leq |y-x|$, or $||x|-|y|| \leq |x-y|$.

15. *Under what conditions does equality hold in the Schwarz inequality?*
Suppose $b_j = ca_j$ for each j and some $c \in \mathbb{C}$. Then if each $a_j = x_j + iy_j$ we have

$$\begin{aligned}
\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 &= \left|\sum_{j=1}^{n} a_j \overline{ca_j}\right|^2 \\
&= |\overline{c}|^2 \left|\sum_{j=1}^{n} a_j \overline{a_j}\right|^2 \\
&= |c|^2 \left|\sum_{j=1}^{n} \left(x_j^2 + y_j^2\right)\right|^2 \\
&= |c|^2 \left[\sum_{j=1}^{n} \left(x_j^2 + y_j^2\right)\right]^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{n} \left|a_j\right|^2 \sum_{j=1}^{n} \left|b_j\right|^2 &= \sum_{j=1}^{n} \left|a_j\right|^2 \sum_{j=1}^{n} \left|ca_j\right|^2 \\
&= |c|^2 \sum_{j=1}^{n} \left|a_j\right|^2 \sum_{j=1}^{n} \left|a_j\right|^2 \\
&= |c|^2 \left[\sum_{j=1}^{n} \left|a_j\right|^2\right]^2 \\
&= |c|^2 \left[\sum_{j=1}^{n} \left(x_j^2 + y_j^2\right)\right]^2 \\
&= \left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2.
\end{aligned}$$

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7. *Prove that if $0 < a < b$, then*

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \leq \frac{a+b}{2}$ holds for all $a, b \geq 0$.

We'll work left to right. The first inequality is derived as follows:

$$\begin{aligned} a &< b \\ a^2 &< ab \\ a &< \sqrt{ab}. \end{aligned}$$

For the next step, we can do this:

$$\begin{aligned} (a-b)^2 &> 0 \\ a^2-2ab+b^2 &> 0 \\ a^2+2ab+b^2 &> 4ab \\ (a+b)^2 &> \left(2\sqrt{ab}\right)^2 \\ a+b &> 2\sqrt{ab} \\ \frac{a+b}{2} &> \sqrt{ab}. \end{aligned}$$

The last step is the easy one:

$$\begin{aligned} a &< b \\ \frac{a}{2} &< \frac{b}{2} \\ \frac{a}{2}+\frac{b}{2} &< \frac{b}{2}+\frac{b}{2} \\ \frac{a+b}{2} &< b. \end{aligned}$$

Finally, stringing them all together gives us the inequalities we want.

13. *The maximum of two numbers x and y is denoted by* $\max(x,y)$. *Thus* $\max(-1,3) = \max(3,3) = 3$ *and* $\max(-1,-4) = \max(-4,-1) = -1$. *The minimum of x and y is denoted by* $\min(x,y)$. *Prove that*

$$\max(x,y) = \frac{x+y+|y-x|}{2},$$

$$\min(x,y) = \frac{x+y-|y-x|}{2}.$$

Derive a formula for $\max(x,y,z)$ *and* $\min(x,y,z)$*, using, for example,*

$$\max(x,y,z) = \max(x,\max(y,z)).$$

Suppose $x \le y$. Then $\max(x,y) = y$ and $\min(x,y) = x$; also $|y-x| = y-x$. Thus we have

$$\begin{aligned}\frac{x+y+|y-x|}{2} &= \frac{x+y+y-x}{2} \\ &= \frac{2y}{2} \\ &= y \\ &= \max(y,x)\end{aligned}$$

and

$$\begin{aligned}\frac{x+y-|y-x|}{2} &= \frac{x+y-y+x}{2} \\ &= \frac{2x}{2} \\ &= x \\ &= \min(y,x).\end{aligned}$$

To derive $\max(x,y,z)$ using $\max(x,\max(y,z))$, we simply use two applications of the max formula and simplify:

$$\begin{aligned}\max(x,y,z) &= \max(x,\max(y,z)) \\ &= \frac{x+\max(y,z)+|\max(y,z)-x|}{2} \\ &= \frac{x+\frac{y+z+|y-z|}{2}+\left|\frac{y+z+|y-z|}{2}-x\right|}{2} \\ &= \frac{x+\frac{1}{2}(y+z+|y-z|)+\frac{1}{2}|y+z-2x+|y-z||}{2} \\ &= \frac{1}{2}x+\frac{1}{4}(y+z+|y-z|)+\frac{1}{4}|y+z-2x+|y-z||.\end{aligned}$$

$\min(x,y,z)$ can be done in the same manner:

$$\begin{aligned}\max(x,y,z) &= \min(x,\min(y,z)) \\ &= \frac{x+\min(y,z)-|\min(y,z)-x|}{2} \\ &= \frac{x+\frac{y+z-|y-z|}{2}-\left|\frac{y+z-|y-z|}{2}-x\right|}{2} \\ &= \frac{x+\frac{1}{2}(y+z-|y-z|)-\frac{1}{2}|y+z-2x-|y-z||}{2} \\ &= \frac{1}{2}x+\frac{1}{4}(y+z-|y-z|)-\frac{1}{4}|y+z-2x-|y-z||.\end{aligned}$$

14.

(a) *Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \ge 0$. Why is it then obvious for $a \le 0$?)*
All right, we'll start with the case $a \ge 0$. Then $-a \le 0$, and so by the definition of absolute value $|a| = a$ and $|-a| = -(-a) = a = |a|$.
If $a \le 0$ then $-a \ge 0$, so $|a| = -a$ and $|-a| = -a$, and so we're done.

(b) *Prove that $-b \le a \le b$ if and only if $|a| \le b$. In particular, it follows that $-|a| \le a \le |a|$.*
$\Rightarrow$: Suppose that $-b \le a \le b$. If $a \ge 0$ then $a = |a| \le b$. If $a < 0$ then $|a| = -a \Rightarrow a = -|a| \le b$, so $|a| \ge -b$.
$\Leftarrow$: Suppose that $|a| \le b$. Since $a \le |a|$, we have $a \le |a| \le b$ and $a \le b$ by transitivity. But we also have $-a \le |-a|$ and since we showed that $|-a| = |a|$ in part (a) we have $-a \le |a| \le b$, leading to $-a \le b$ and $-b \le a$. Stringing these together gives $-b \le a \le b$.

(c) *Use this fact to give a new proof that* $|a+b| \leq |a|+|b|$.
We'll do three cases.
Since we know $a \leq |a| \Rightarrow -|a| \leq a \leq |a|$ and $b \leq |b| \Rightarrow -|b| \leq b \leq |b|$, we can write $-|a|-|b| \leq a+b \leq |a|+|b|$, or $-(|a|+|b|) \leq a+b \leq |a|+|b|$. Then the above property applies: $|a+b| \leq |a|+|b|$.

20. *Prove that if*

$$|x-x_0| < \frac{\varepsilon}{2} \text{ and } |y-y_0| < \frac{\varepsilon}{2},$$

then

$$\begin{aligned} |(x+y)-(x_0+y_0)| &< \varepsilon, \\ |(x-y)-(x_0-y_0)| &< \varepsilon. \end{aligned}$$

Both of these can be done witih some rearrangement and the Triangle Inequality:

$$\begin{aligned} |(x+y)-(x_0+y_0)| &= |(x-x_0)+(y-y_0)| \\ &\leq |x-x_0|+|y-y_0| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

$$\begin{aligned} |(x-y)-(x_0-y_0)| &= |(x-x_0)+(-y+y_0)| \\ &\leq |x-x_0|+|-y+y_0| \\ &= |x-x_0|+|y-y_0| \\ &\leq \varepsilon \text{ by above.} \end{aligned}$$

22. *Prove that if* $y_0 \neq 0$ *and*

$$|y-y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right),$$

then $y \neq 0$ *and*

$$\left|\frac{1}{y}-\frac{1}{y_0}\right| < \varepsilon.$$

First, let's look at why y cannot be zero. Suppose it is. Then $|y-y_0| = |y_0|$, which is supposed to be less than the smaller of $\frac{|y_0|}{2}$ and $\frac{\varepsilon|y_0|^2}{2}$. Obviously $|y_0|$ can't be smaller than $\frac{|y_0|}{2}$.
We can assume $y \neq 0$ because if it was then $|y_0| < \frac{|y_0|}{2}$

- First, $|y_0| < \frac{\varepsilon|y_0|^2}{2}$. All these terms are in $\mathbb{R}^+$, so after some manipulation we come to $\frac{2}{\varepsilon} < |y_0|$
- Second, $\frac{\varepsilon|y_0|^2}{2} < \frac{|y_0|}{2}$, which leads to $|y_0| < \frac{1}{\varepsilon}$

Putting these together leads to $\frac{2}{\varepsilon} < \frac{1}{\varepsilon}$, which is impossible if $\varepsilon \in \mathbb{R}^+$. This contradiction thus implies that $y \neq 0$. Whew!
Let $|y-y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right)$. Then $|y-y_0| < \frac{|y_0|}{2}$ and $|y-y_0| < \frac{\varepsilon|y_0|^2}{2}$. Since $|y_0|-|y| < |y-y_0|$ we have $|y_0|-|y| < \frac{|y_0|}{2}$ by transitivity, leading to $\frac{|y_0|}{2} < |y|$
23. *Replace the question marks in the following statement by expressions involving* ε, x_0 *and* y_0 *so that the conclusion will be true:*
If $y_0 \neq 0$ *and*

$$|y-y_0| < ? \text{ and } |x-x_0| < ?$$

then $y \neq 0$ and

$$\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon.$$

This problem is trivial in the sense that its solution follows from Problems 21 and 22 with almost no work at all (notice that $\frac{x}{y} = x \cdot \frac{1}{y}$). The crucial point is not to become confused; decide which of the two problems should be used first, and don't panic if your answer looks unlikely.

If we substutte $\frac{1}{y}$ for y and $\frac{1}{y_0}$ for y_0 then Problem 21 states that if

$$|x - x_0| < \min\left(\frac{1}{2} \cdot \frac{\varepsilon\,|y_0|}{1+|y_0|}, 1\right) \text{ and } \left|\frac{1}{y} - \frac{1}{y_0}\right| < \frac{\varepsilon}{2\,|x_0|+1}$$

then $\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon$.

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2. *Find a formula for*

(a) $\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2n-1)$.
I suspect that this adds up to n^2. I'll work by induction.

- The base case is easy as usual: 1^2 is indeed 1.
- Suppose $1+3+\cdots+(2n-1) = n^2$. Then $1+3+\cdots+(2n-1)+[2(n+1)-1] = n^2+(2n+1) = (n+1)^2$.

(b) $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2$.
Note that if we add all the even numbers to the original sum, we get the sum of the squares of all the numbers from 1 to $2n$. With this we can deduce that

$$\begin{aligned}
\left[1^2+3^2+5^2+\cdots+(2n-1)^2\right]+\left[2^2+4^2+\cdots+(2n)^2\right] &= \sum_{i=1}^{2n} n^2 \\
\left[1^2+3^2+5^2+\cdots+(2n-1)^2\right]+4\left[1^2+2^2+\cdots+n^2\right] &= \sum_{i=1}^{2n} n^2 \\
\left[1^2+3^2+5^2+\cdots+(2n-1)^2\right] &= \sum_{i=1}^{2n} n^2 - 4\sum_{i=1}^{n} n^2 \\
\left[1^2+3^2+5^2+\cdots+(2n-1)^2\right] &= \frac{2n(2n+1)(4n+1)}{6} - 4\frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(2n+1)(2n-1)}{3}
\end{aligned}$$

Hint: what do these expressions have to do with $1+2+3+\cdots+2n$ and $1^2+2^2+3^2+\cdots+(2n)^2$?

3. *If $0 \le k \le n$, the "binomial coefficient" $\binom{n}{k}$ is defined by*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \text{ if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$

- *(a) Prove that*

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(The proof does not require an induction argument.)
I'll work this backwards.

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!\,[n-(k-1)]!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{k \cdot n!}{(n-k+1)\,k!\,(n-k)!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{n!}{k!\,(n-k)!}\left(\frac{k}{n+1-k} + 1\right) \\
&= \frac{n!}{k!\,(n-k)!}\left(\frac{n+1}{n+1-k}\right) \\
&= \frac{(n+1)!}{k!\,[(n+1)-k]!} \\
&= \binom{n+1}{k}
\end{aligned}$$

- *(d) Prove the "binomial theorem": If a and b are any numbers and n is a natural number, then*

$$\begin{aligned}
(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \\
&= \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j.
\end{aligned}$$

Can do for $n = 1$ easy
Suppose

$$(a+b)^{n-1} = a^{n-1} + \binom{n-1}{1}a^{n-1-1}b + \binom{n-2}{2}a^{n-1-2}b^2 + \cdots + \binom{n-1}{n-1-1}ab^{n-1-1} + b^{n-1}.$$

Then multiplying both sides by $a+b$ yields

$$\begin{aligned}
(a+b)^{n-1} &= (a+b)\left[a^{n-1} + \binom{n-1}{1}a^{n-1-1}b + \binom{n-2}{2}a^{n-1-2}b^2 + \cdots + \binom{n-1}{n-1-1}ab^{n-1-1} + b^{n-1}\right] \\
&= a\left[a^{n-1} + \binom{n-1}{1}a^{n-1-1}b + \binom{n-2}{2}a^{n-1-2}b^2 + \cdots + \binom{n-1}{n-1-1}ab^{n-1-1} + b^{n-1}\right] \\
&\quad + b\left[a^{n-1} + \binom{n-1}{1}a^{n-1-1}b + \binom{n-2}{2}a^{n-1-2}b^2 + \cdots + \binom{n-1}{n-1-1}ab^{n-1-1} + b^{n-1}\right] \\
&= a^n + \binom{n-1}{1}a^{n-1}b + \binom{n-2}{2}a^{n-2}b^2 + \cdots + \binom{n-1}{n-1-1}ab^{n-1} + = b^{n-1}a \\
&\quad + 1a^{n-1}b + \binom{n-1}{1}a^{n-1}b^2 + \binom{n-2}{1}a^{n-2}b^3 + \cdots + \binom{n-1}{n-2}ab^{n-1} + b^n \\
&= a^n + \left[\binom{n-1}{1} + \binom{n-1}{1}\right]
\end{aligned}$$

5.

(a) *Prove by induction on n that*

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

if $r \neq 1$.
The case for $n = 1$ is easy enough:

$$\begin{aligned}\frac{1-r^2}{1-r} &= \frac{(1-r)(1+r)}{1-r} \\ &= 1+r.\end{aligned}$$

Suppose that

$$1+r+r^2+\cdots+r^{n-1} = \frac{1-r^n}{1-r}.$$

Then

$$\begin{aligned}1+r+r^2+\cdots+r^{n-1}+r^n &= \frac{1-r^n}{1-r}+r^n \\ &= \frac{1-r^n+r^n-r^{n+1}}{1-r} \\ &= \frac{1-r^{n+1}}{1-r}.\end{aligned}$$

(b) *Derive this result by setting* $S = 1+r+\cdots+r^n$*, multiplying this equation by* r*, and solving the two equations for* S*.*

$$\begin{aligned}S &= 1+r+\cdots+r^n \\ rS &= r+r^2+\cdots+r^{n+1}.\end{aligned}$$

Subtracting the second equation from the first yields

$$\begin{aligned}S-rS &= 1+r+\cdots+r^n-r-r^2-\cdots-r^{n+1} \\ S(1-r) &= 1-r^{n+1} \\ S &= \frac{1-r^{n+1}}{1-r}.\end{aligned}$$

6. *The formula for* $1^2+\cdots+n^2$ *may be derived as follows. We begin with the formula*

$$(k+1)^3-k^3 = 3k^2+3k+1.$$

Writing this formula for $k = 1,\ldots,n$ *and adding, we obtain*

$$\begin{aligned}2^3-1^3 &= 3\cdot 1^2+3\cdot 1+1 \\ 3^3-2^3 &= 3\cdot 2^2+3\cdot 2+1 \\ &\vdots\end{aligned}$$

$$\frac{(n+1)^3-n^3 = 3\cdot n^2+3\cdot n+1}{(n+1)^3-1 = 3\,[1^2+\cdots+n^2]+3\,[1+\cdots+n]+n.}$$

Thus we can find $\sum_{k=1}^{n} k^2$ *if we already know* $\sum_{k=1}^{n} k$ *(which could have been found in a similar way). Use this method to find*

(a) $1^3+\cdots+n^3$.

We start with

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 6k^2 + 4k + 1$$

for $k = 1, \dots, n$. From this we have

$$\begin{aligned}
(n+1)^4 - 1 &= 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n \\
\sum_{k=1}^{n} k^3 &= \frac{1}{4}\left[(n+1)^4 - 1 - 6\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k - n\right] \\
&= \frac{1}{4}\left[(n+1)^4 - 1 - 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} - n\right]
\end{aligned}$$

which after a whole lot of algebra simplifies to $\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$.

(b) $1^4 + \cdots + n^4$.
Should be similar to the last one.

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

and so

$$\begin{aligned}
(n+1)^5 - 1 &= 5\sum_{k=1}^{n} k^4 + 10\sum_{k=1}^{n} k^3 + 10\sum_{k=1}^{n} k^2 + 5\sum_{k=1}^{n} k + n \\
\sum_{k=1}^{n} k^4 &= \frac{1}{5}\left[(n+1)^5 - 1 - 10\sum_{k=1}^{n} k^3 - 10\sum_{k=1}^{n} k^2 - 5\sum_{k=1}^{n} k - n\right] \\
&= \frac{1}{5}\left[(n+1)^5 - 1 - 10\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) - 10\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) - 5\left(\frac{n^2}{2} + \frac{n}{2}\right) - n\right]
\end{aligned}$$

and again after all the manipulation is done we get $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

Handout

3. *Suppose that the number a has the property that for every natural number n, $a \le \frac{1}{n}$. Prove that $a \le 0$.*

Suppose that $a > 0$. By the Archemedian Principle there exists $n \in \mathbb{N}$ such that $na > 1$. But this means that $a > \frac{1}{n}$. Thus we have the following: $a > 0$ implies that there exists $n \in \mathbb{N}$ such that $a > \frac{1}{n}$, which is the contraposition of what we wish to prove.

4. *Given a real number a, define $S = \{x \mid x \in \mathbb{Q}, x < a\}$. Prove that $a = \sup S$.*

We know that S is bounded above by a because that's how we defined it. Since S is a set of real numbers, we know that $\sup S$ exists as a real number. But what if some other real number, call it b, is the supremum of S? We'll look at two cases.

- If $b > a$, then there exists $q \in S$ such that $a < q \le b$. But this is impossible since a is an upper bound of S.
- If $b < a$ then let $b' = \frac{a-b}{2}$ (note that $b < b'$). Then there exists $q \in \mathbb{Q}$ such that $b' \le q < a$. This again proves to be trouble because it means that $b < q$ and b is no longer an upper bound for S (b must be an upper bound of S if it is to be the supremum). Thus this case is also impossible.

Therefore by trichotomy we must have $b = a$, and so $a = \sup S$.

5. *Prove that every bounded, nonempty set of natural numbers has a maximum.*

First, note that since $\mathbb{N}$ is bounded it has a finite number of elements. We'll proceed by induction. Let $S_n = \{a_1, a_2, \dots, a_n\}$ be a bounded set of natural numbers with n elements. Obviously, $S_1 = \{a_1\}$ has a maximum since a_1 is the only thing in it. Suppose $S_k = \{a_1, a_2, \dots, a_k\}$ has a maximum α, and consider $S_{k+1} = S_k \cup \{a_{k+1}\}$. If $\alpha \geq a_{k+1}$, then it is the maximum of S_{k+1}; if not then a_{k+1} is the maximum. Either way S_{k+1} has a maximum, and so by induction the premise holds true.

6. *Let A and B be two sets of real numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let C (E) be the set of all sums (products) of the form $x + y$ (xy), where $x \in A$ and $y \in B$. Is it necessarily true that $a + b = \sup C$ ($ab = \sup E$)? Prove or disprove.*

- $a + b = \sup C$: Let $c = a + b$. We first need to show that c is an upper bound for C. Let $x \in C$. Then $x = a' + b'$, where $a' \in A$ and $b' \in B$. This implies that $a' \leq a$ and $b' \leq b$. But this means $a' + b' \leq a + b$, implying $x \leq c$ by transitivity.
  We now know that c is an upper bound, but still need to show it is the *least* upper bound. Suppose it is not; that is, suppose $N = \sup\{C\}$ and that $c > N$. Then $N = c - \alpha$ for some positive α. Thus if we let $\varepsilon = \frac{\alpha}{4}$ there exists an element $a_1 \in a$ such that $a_1 \geq a - \varepsilon$. Similarly, $b_1 \geq b - \varepsilon$ for some $b_1 \in B$. This implies $a_1 + b_1 \geq a + b - 2\varepsilon = c - \frac{\alpha}{4}$. But $a + b \in S$, and $c - \frac{\alpha}{2} > c - \alpha = N$, and so we have found an element in C that is greater than N; contradicting the assumption that N is an upper bound. Therefore the upper bound $N < c$ cannot exist, and so c must be the least upper bound.

- $ab = \sup E$: This turns out to be *not* true in general. Let $a = (-\infty, 0)$ and $b = \{-1\}$ then $E = (0, \infty)$, which clearly lacks a supremum.

7. *(The monotone property for suprema) Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$. If B has a supremum, prove that* $\sup A$ *exists and* $\sup A \leq \sup B$. *Formulate and prove a similar statement for infima.*

8. *Suppose $A, B \subseteq \mathbb{R}$ and $E = A \cup B$. If E has a supremum and both A and B are nonempty, prove that* $\sup A$ *and* $\sup B$ *both exist, and* $\sup E$ *is one of the numbers of* $\sup A$ *and* $\sup B$. *What can you say about* $\inf(A \cup B)$? $\sup(A \cap B)$? $\inf(A \cap B)$? *(Assume $A \cap B \neq \varnothing$). Must support your assertion.*

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