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Z5. Let S be a nonempty set of natural numbers that has an upper bound. By the Completeness axiom, $\alpha = \sup S$ exists (but isn't necessarily an integer). By the approximate property of supremums, $\exists \mathbb{N} \in S$ such that $\alpha - \frac{1}{2} < N \leq \alpha$ ($\varepsilon = \frac{1}{2}$ in the approx. property). Now we show that $n \leq N$ for all $n \in S$; then N will be a maximum and equal to α.)

If some $n_0 \in S$ such that $n_0 > N$, then $n_0 \geq N+1 = N + \frac{1}{2} + \frac{1}{2} > \alpha + \frac{1}{2}$ which is a contradiction.

can't say it's a finite set w/o proof, likely cyclical; if we can prove then one point back.

R15. Equality holds in the Schwarz inequality iff either one of the vectors is the zero vector or $\exists r \in \mathbb{R}$, $r \neq 0$ st $\mathbf{x} = r\mathbf{y}$. We calculate

$$\sum_{i=1}^{n} (x_i - ry_i)^2 = r^2 \left(\sum_{i=1}^{n} y_i^2 \right) - 2r \left(\sum_{i=1}^{n} x_i y_i \right) + \left(\sum_{i=1}^{n} x_i^2 \right).$$

If we let the right side be equal to zero, we have a quadratic in r. The discriminant is

$$D = \left(\sum_{i=1}^{n} x_i y_i \right)^2 - \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right).$$

If equality holds then $D = 0$. Thus the quadratic equation of r has a (repeated) solution, say r_0. Then $\mathbf{x} = r_0 \mathbf{y}$.

Theorem $\mathbb{Q}$ is countable.

Proof 1. Let $A_0 = \{0\}$, $A_1 = \left\{ \frac{-1}{1}, \frac{1}{1}, \frac{-2}{1}, \frac{2}{1}, \ldots \right\}$, $A_2 = \left\{ \frac{-1}{2}, \frac{1}{2}, \frac{-2}{2}, \frac{2}{2}, \ldots \right\}$ and so on. Then $\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$. This is a countable union of countable sets, therefore $\mathbb{Q}$ is countable. (Note that we can make them all disjoint if we require that $(p,q) = 1$.

Proof 2. Consider $\mathbb{Q}^+$. Let $r = \frac{p}{q}$ in lowest terms. Define $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ by $f(r) = (p,q)$. Note that f is into. It's also one-to-one, because $\frac{p}{q} = \frac{r}{s}$ iff $p = r$ and $q = s$ if they're relatively prime. Now $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$, which is the union of three countable sets; so $\mathbb{Q}$ is countable.

Combining Countable Sets

(a) Every subset of a countable set is again countable or finite.

(b) The set of all ordered pairs of integers is countable

(c) The countable union of countable sets is countable.

(d) The finite cross product of countable sets is countable.

Basic countable sets:

- $\mathbb{Q}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.
- all the circles on the plane w/ rational radius and center (cause it's just $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$)
- The set of disjoint circles

Theorem Let S be the set of all sequences whose elements are the digits 0 and 1. This set S is not countable. The elements of S are called **0-1 sequences**.

Proof. (Cantor's Diagonal Process)

(a) We first show, using Cantor's diagonal process, that every countable subset of of S must be a proper subset of S. Let A be a countable subset of S. Then A can be enumerated as $a_{11}, a_{12}, a_{13}, \ldots, a_{21}, a_{22}, \ldots$ etc. with $a_{ij} \in \{0,1\}$. Construct $a = a_{11}^C, a_{22}^C, a_{33}^C, \ldots$ where a_{ij}^C is $1 - a_{ij}$. Then $a_{11}^C \neq a_{11}$, $a_{22}^C \neq a_{22}$, etc. Then a isn't on the list; it must be a new element of S that's not in A. Thus A is a proper subset of S.

(b) S is uncountable.
If S is countable, then according to (a), S has to be a proper subset of S. This is silly, so S is uncountable.

Can this be generalized to $[0,1]$? The proof of Homework 14 has something to do with this; use Cantor's there

Theorem. $[0,1)$ is uncountable.

Proof.

(a) Any number $x \in [0,1)$ has a binary expansion:

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k},$$

where $x_k \in \{0,1\}$.

(b) Some numbers may have two binary expansions. For example, $0.10\ldots = 0.01\overline{1}$ (in the same way that $.99999\ldots = 1$); one terminates in ones and the other in zeros. Let E be the set of all binary expansions that terminate in ones. Then we can show that E is countable (it's the countable union of finite sets)

(c) Let S be the set of 0-1 sequences. This set clearly has a one-to-one correspondence with A, the set of all binary expansions where the double-sequences in (2) are counted separately. We proved last theorem that S is uncountable. Therefore A is uncountable. Thus $A \backslash E$ is uncountable. But we just removed the duplicates.

(d) Therefore, there is a 1-1 and onto function from $[0,1) \to A \backslash E$ because each $x \in [0,1)$ has a unique binary expansion that does not terminate in ones. This means it's equivalent to $A \backslash E$, which is uncountable.

test will cover up through uncountables.